GEOMETRY

BOOK 2
(TEXT)

FOR
CLASS 6

(Experimental Edition)

Prepared by
NCERT Mathematics Study Group
(Central College, Bangalore University)

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PREFACE

For the last ten or twelve years a revolution, popularly known as the 'New Mathematics Revolution', has swept the world. It has involved an expenditure of millions of dollars, roubles, pounds, francs, and rupees. What is more important, it has involved millions of man-hours of school teachers, college teachers, educational administrators and some of the foremost mathematicians of modern times. It has changed the attitude to mathematics of billions of children, teachers and parents

About a hundred projects* have been set up all over the world for developing new curricular materials including text-books, teachers' guides, charts, models, film strips, films, T. V. lessons etc. and for training teachers in new ideas. In India, the efforts for improving school mathematics have included setting up a panel of distinguished mathematicians under the chairmanship of Prof. Ram Behari, an editorial board under my chief editorship, projects under the auspices of the department of science education of the NCERT, organisation of six study groups at Bangalore, Baroda, Delhi, Jaipur, Jadavpur and Kanpur, a large number of summer institutes for training of teachers and a national conference on school mathematics.**

The six study groups for developing new curricular materials in mathematics were set up in 1966 as a result of the decisions of the conference on mathematics and science education held under the chairmanship of Prof. D.S Kothari. The co-ordinating committee of the mathematics Study Groups decided to assign responsibilities as follows: Primary Mathematics (Classes 1-4) and Applications of Mathematics to the Kanpur Study Group, Algebra (including Arithmetic) for Classes 5 to 7 and 8 to 10 to the Baroda, New Delhi and Jaipur Groups and Geometry (including Co-ordinate Geometry and Trigonometry) for Classes 5 to 7 and 8 to 10 to the Bangalore and Jadavpur Groups. The entire work was to be co-ordinated by the co-ordinating committee consisting of the six Directors of the Study Groups.

So far the Study Groups have produced the following books:

(i) Handbooks for Teachers of Classes 1 and 2
(ii) Algebra text-books and their teachers' guides for Classes 5, 6 and 7
(iii) Geometry text-books and their teachers' guides for Classes 5, 6 and 7.

Experimental editions of these books are now being made avail able for try-outs in some selected schools. The final editions of these books improved in the light of the reactions of students and teachers will be made available for open use by state governments sometime in 1970. The present editions are for restricted use by authorised schools only.

Though Baroda, Delhi and Jaipur Groups worked jointly in writing the earlier versions of algebra books, the final version was prepared by the Jaipur Group. Throughout the preparation of the final version, the Jaipur Group was assisted by Shri. P. K. Srinivasan of the Kanpur Study Group.

The final version of the geometry text-books has been prepared by the Bangalore Group and that of the geometry teachers' guides has been prepared by the Jadavpur Group.

In algebra, the approach has been through sets, truth sets of open sentences including inequalities and structures of the systems of natural numbers, integers and rational numbers. In geometry, the approach has been through transformations and symmetries. Both the approaches are likely to be new to teachers and therefore all efforts will be made to train the teachers in the new ideas. Even when training programmes are not available, it is hoped that the teachers' handbooks will be sufficiently lucid and clear to provide sufficient help to interested teachers.

Most of the ideas contained in these books have been tried out with many batches of students and the reactio nof students and teachers has been one of unmitigated enthusiasm. These were also discussed at the National Workshop on School Mathematics held at IIT Kanpur in December, 1968 and attended by representatives of eleven state institutes of education. Their enthusiastic support has been a source of strength to us.

I must take this opportunity of thanking Prof. D. S. Kothari, Chairman of the co-ordinating committee of all Science and Mathematics Study Groups, Shri. L. S. Chandrakant, former Joint Director, NCERT, Dr. S. K. Mitra, present Joint Director, NCERT, Dr. M. C. Pant, Head of the Department of Science Education, NCERT, Shri. Rajinder Prasad, Field Officer of the NCERT, Shri. R. C. Sharma, Reader in Mathematics, Shri D. Raghavan, Chief Publication Unit, NCERT, Shri. Chakravarty, Chief Production Officer, NCERT, Shri. M. A. Srirama of Bangalore Government Press for administrative support at all levels without which production and printing of these books would not have been possible.

On the academic side, I would like to thank all the Directors and members of the study groups whose names are given elsewhere, for their co-operation. I must make special mention of Prof. K. Venkatachaliengar whose group prepared the first detailed curriculum of Geometry and the first version of the geometry books, of Prof. G C. Patni, for giving valuable leadership to the Jaipur Group during the preparation of the final version of algebra books and of Dr. D. K. Sinha for getting Geometry teachers' guides prepared.

I must also thank the managers of the presses for their sincere co-operation.

All the teachers are requested to fill in the proformas given in the teachers' guides and return them to the undersigned.

J. N. KAPUR
Convener, NCERT
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I. I. T. Kanpur.

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* Lockard, D. Report of the International Clearing House on Curricular Development in Science Education, Maryland University, U.S A.

** Kapur, J.N. (Editor) Proceedings of the National Conference on School Mathematics, Mathematical Association of India.

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ERRATA

Page	*Line*	*For*	*Read*
69	6	$\mid PQ \mid$	$\mid XY \mid$
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74	7	DEF	$\angle$DEF
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CONTENTS

Separation (Order) Relation

You have already studied the separation relations on a line. In this chapter, you will study the separation properties of a plane, regions in a plane and properties that characterise a region. These will be illustrated with interesting examples

1.1 Separation

Consider the set of points on a line l. Any two points A and B on l are connected naturally by a line segment joining them which belongs entirely to the set of points on the line l.

Suppose a point P such that $A < P < B$ on l is removed (Fig. 1). Then the remaining set M of points of l consists of two mutually disjoint parts viz., the two *open* rays PA and PB (open because their common initial point P is removed).

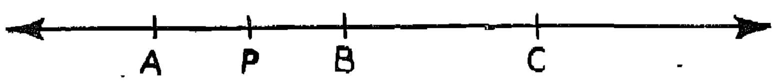


Fig. 1.

That is, A and B cannot be connected by a line segment which belongs entirely to M since the segment AB contains the point P which does not belong to M. Therefore we say that M is not a connected set. Hence removal of a point from a line affects the connectivity of the line.

Next, suppose we remove two points P and Q from l (Fig. 2).

Fig 2.

The remaining set of points forms three distinct, mutually disjoint parts, namely, the two *open* rays PA and QB and the *open* segment PQ.

In the same way consider the set of all points in a plane. Any two points of this set can be connected by a segment belonging entirely to the set. But if we remove from the plane, the set of points on a line l, then the set N of the remaining points is made up of two disjoint parts. If A and B are two points on the opposite sides H_1 and H_2 of the line (Fig. 3),

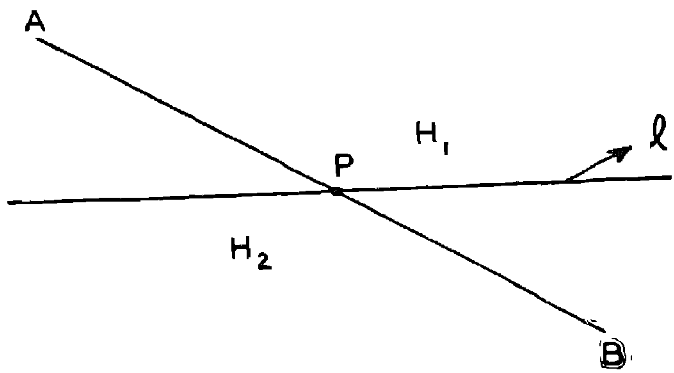


Fig. 3.

the segment AB contains a point P of l, that is, not belonging to the set N. Therefore the segment AB does not belong entirely to N. In fact no path in the plane connecting A and B belongs to the set N because it always contains a point of l where it crosses the line l. So we say

> A line divides the plane into two regions which are called half planes.

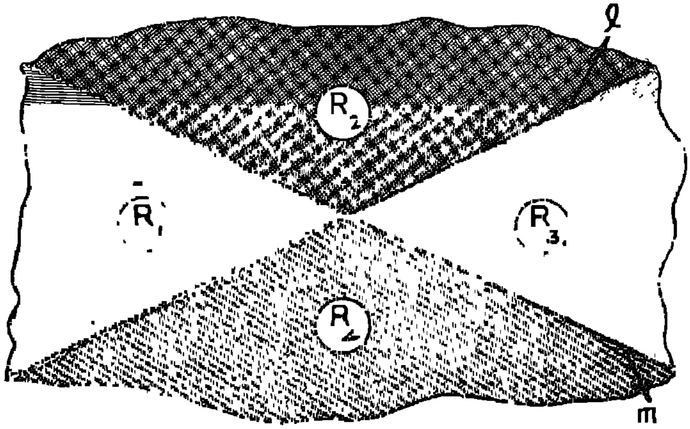


Fig. 4.

If we remove from the plane the set of points on two intersecting lines l and m (Fig. 4), we get four regions R_1, R_2, R_3 and R_4, that is

Two intersecting lines divide the plane into four regions.

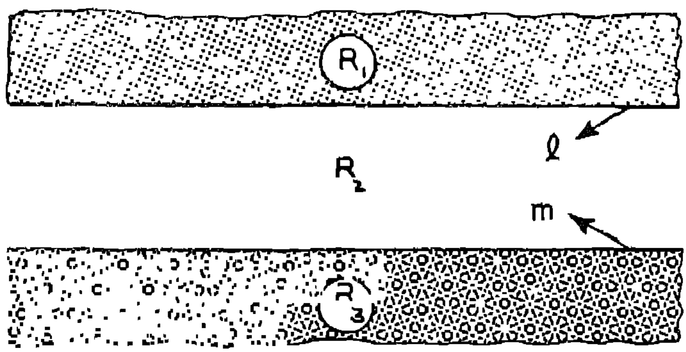


Fig. 5.

Figure 5 shows that

Two parallel lines divide the plane into three regions.

If we remove from the plane the set of points on a circle C (Fig. 6), the remaining points form two disjoint sets. The points inside

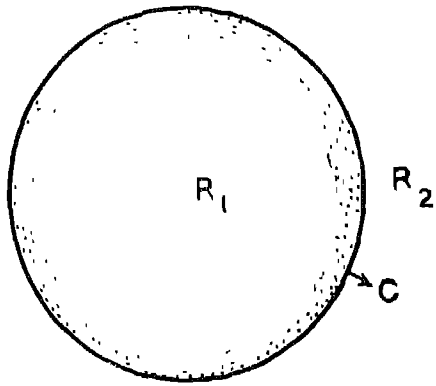


Fig 6.

the circle form *the inner region* or the *inner disc region* R_1 and the set of all points outside the circle forms the *outer region* R_2. That is,

> A circle divides the plane into two regions – the inner disc region and the outer region.

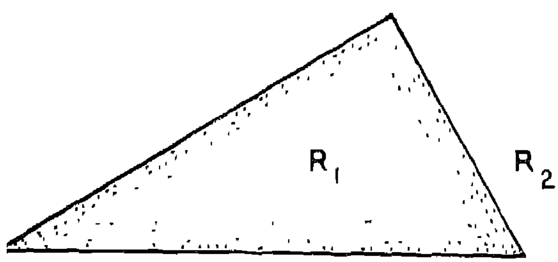


Fig 7.

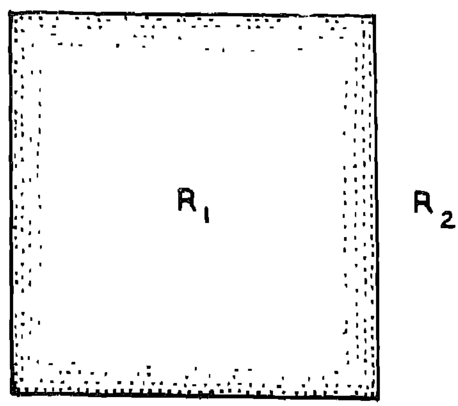


Fig. 8.

In the same way a triangle (Fig. 7), or a square (Fig. 8), or a rectangle (Fig. 9) divides the plane into the inner region R_1 (which

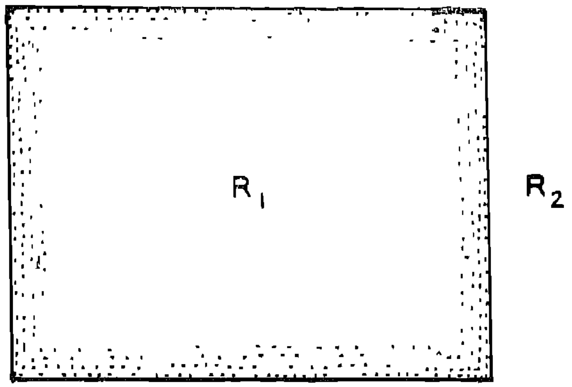


Fig 9.

is shaded) and the outer region R_2. Fig. 10 shows a closed curve C dividing the plane into the inner region R_1 and the outer region R_2.

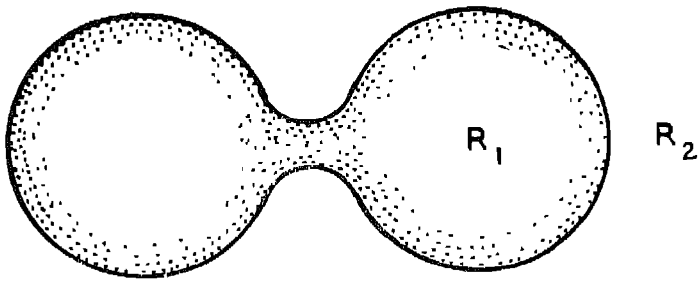
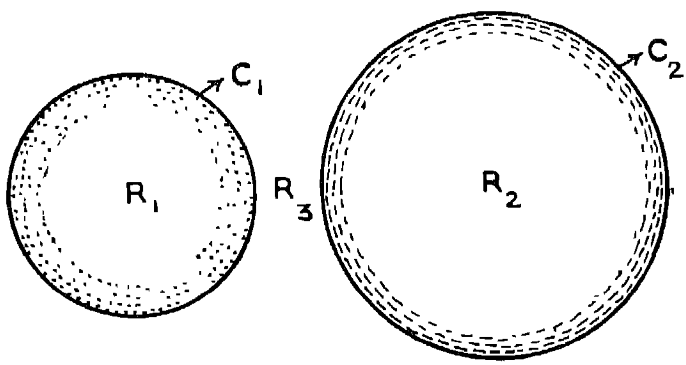


Fig. 10.

Two circles C_1 and C_2, one outside the other (Fig. 11) divide the plane into the 3 regions—R_1 the inner disc region of C_1, R_2 the

Fig. 11.



inner disc region of C_2, and R_3 the set of all points outside both the circles C_1 and C_2.

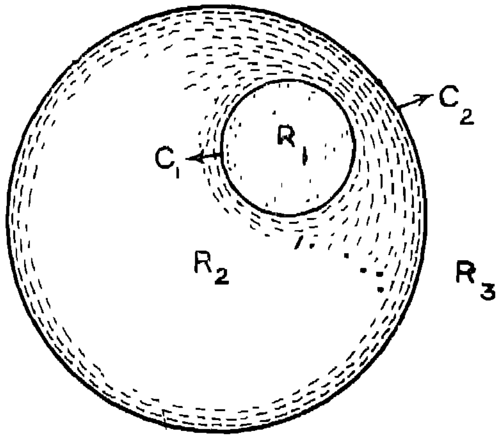


Fig. 12.

Two circles C_1 and C_2 such that C_1 is inside C_2 (Fig. 12), divide the plane into the 3 regions – R_1 the inner disc region of C_1, R_2 the set of points outside C_1 and inside C_2, and R_3 the set of points outside C_2

A fence round a field separates it from the outside land. The region belonging to a cricket or football field is specified by its boundary line.

These examples have given you some idea of a region. We will now understand correctly what is meant by a region

1 2 Connectivity

We can connect any two points of a region by a path (which may consist of a chain of *connected segments*) which runs entirely in the region.

For example, in Fig. 13, the line l divides the plane into the two regions, that is, half-planes H_1 and H_2. A and B belong to H_1; and

the segment path AB also belongs entirely to H_1; C and D belong to H_2; and the segment path CD belongs entirely to H_2. But if P belongs to H_1 and Q to H_2, they do not belong to the same region, because every

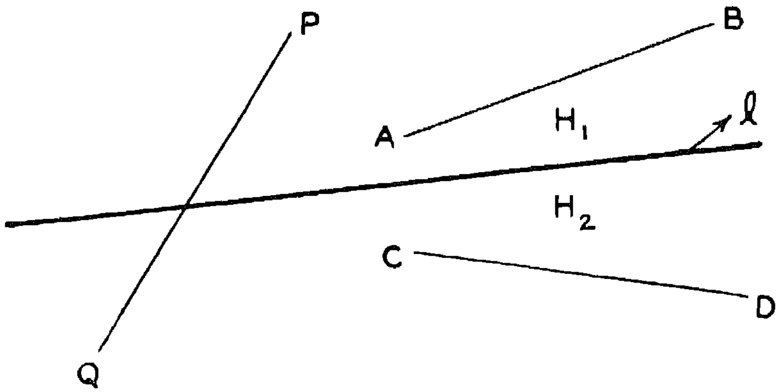


Fig. 13.

segment chain connecting P and Q contains a point R of *l*, that is, a point not belonging either to H_1 or to H_2.

This property is called the *plane separation property.*

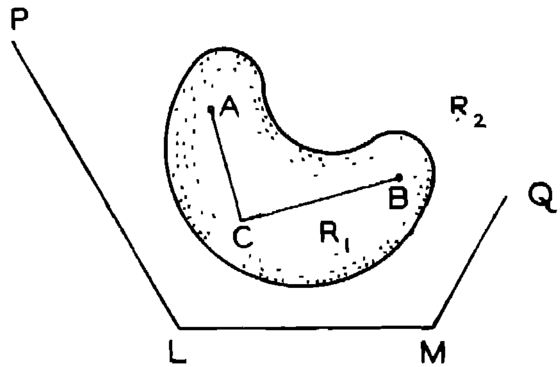


Fig. 14.

In Fig. 14, A and B belong to the inner region R_1, and the segment chain ACB connecting them also belongs entirely to R_1. P and Q belong to R_2 and there is a segment chain PLMQ connecting them and belonging entirely to R_2.

A player in a cricket field or a football field can reach the ball anywhere in the field without crossing the boundary line.

Hence the first property that characterises a region is that:

> The set of points of a region is connected.

Example 1 :—

India (excluding Andaman, Nicobar and other off-shore islands) is an example of a region. We can go from any place in India to any other place in India, say from Bangalore to Delhi, or from

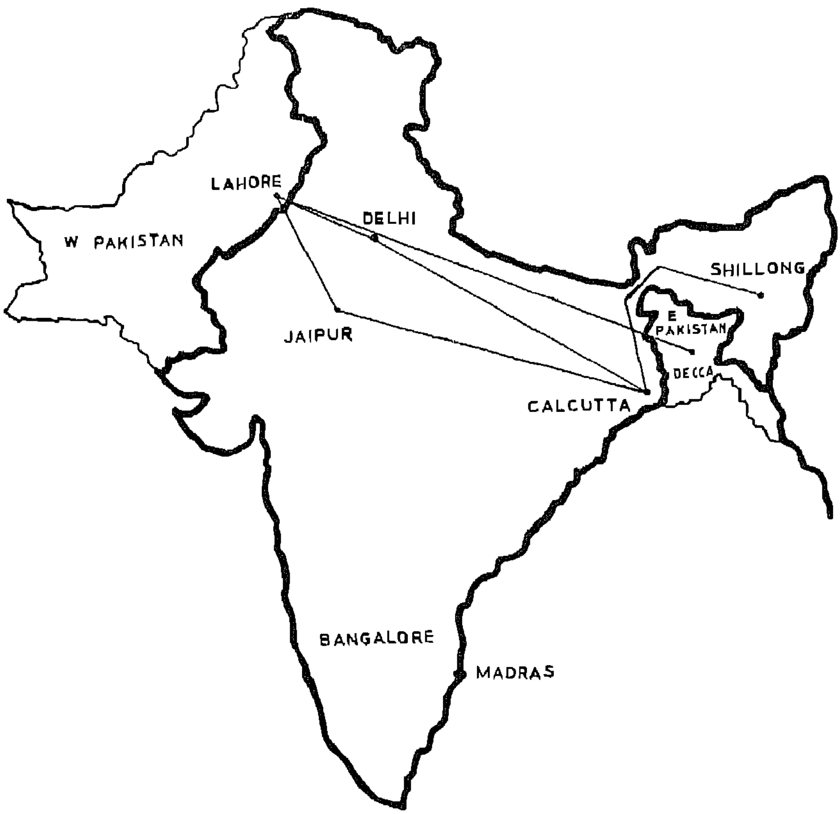


Fig. 15.

Shillong to Calcutta, by a route which lies entirely in India. But it is not possible to make a road from a place in East Pakistan

(say, Dacca) to a place in West Pakistan (say, Lahore), which runs entirely in the country of Pakistan, because Pakistan consists of two separate regions.

Similarly, the five oceans, namely, the Pacific, the Atlantic, the Arctic and the Antarctic oceans together form a connected set and a region. The mainlands of Asia and Europe (excluding England, Japan et cetera and other off-shore islands) together form a region. (Refer to the maps in your Atlas.)

1.3 Openness

Let P be any point of a region R. Then it is always possible to place a sufficiently small dot (the word बिंदि in Hindi or ಬೊಟ್ಟು in Kannada is more suggestive) or a disc round P such that the dot lies entirely in the region R. That is, if P belongs to R, then P is surrounded by a set of nearby (neighbouring) points belonging entirely to R. This is the second property that characterises a region, namely,

> Every point of a region is an inner point.

In other words, every region is an open set.

Let us study some examples which explain this idea.

Example 2 :—

Let H_1 be one of the regions, or half-planes, into which a line l divides the plane. Let P_1 be any point of H_1. Suppose the distance of P_1 from l is 2 kilometers (Fig. 16). Draw a circle with P_1 as centre and *2 km* as radius. This disc, that is, the inner disc region of this circle belongs entirely to H_1. Also any disc round P with radius *less than 2 km* also belongs entirely to H_1. That is, all such discs are the surroundings of P_1, which also belong to H_1.

But suppose P_2 is a point which is at a distance of only 2 meters from l Then the radius of the disc can be taken to be *equal to or less than* 2 meters. If the distance of another point P_3 from l is only 2 cm, the radius of the dot round P_3 can be taken to be *equal to 2 cm* or *less than 2 cm*. Suppose P_4 is still nearer l, at a distance of say, only 0.000001 cm from l, even then we can think of a sufficiently small dot round P_4 of radius equal to or less than 0.000001 cm, and this dot belongs entirely to H_1.

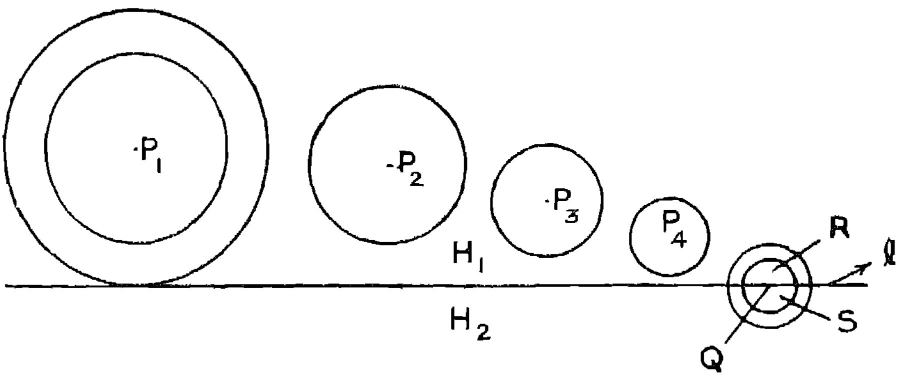


Fig. 16.

Therefore, even when P is a point very close to l, we can take the radius of the dot round P so small that this dot belongs entirely to H_1, so long as P belongs to H_1. That is, every point of H_1 is *its* inner point. Similarly, every point of H_2 is *its* inner point. In other words, if H_1 is coloured red, then every point of H_1 is red, and the colour round *every* point of H_1 is also red.

Example 3 :—

Consider a circle C of radius 3 cm, with centre O. This circle divides the plane into the inner region R_1 and the outer region R_2 (Fig. 17). Let P_1 be a point of R_1, and $|OP_1| = 1.7$ cm.

Then we can put a dot round P_1 whose radius is equal to 1·3 cm or less than 1·3 cm, say 0.8 cm, and this dot surrounding P_1 also belongs entirely to R_1. If P_2 is another point still closer to C such that

$|OP_2| = 2{\cdot}4$ cm, take the radius of the disc round P_2 equal to $0{\cdot}6$ cm or less than $0{\cdot}6$ cm and so on So every point of R_1 is its inner point. Similarly, we can put a sufficiently small disc round a point Q_2 of R_2

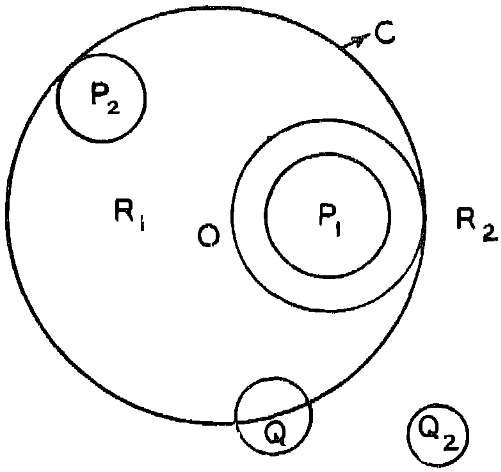


Fig. 17.

such that the disc belongs entirely to R_2. Hence every point of R_2 is its inner point.

Let us now consider examples from some well-known games or the geographical divisions of the earth's surface into countries, states and districts.

Example 4 :—

Most of you are familiar with the game Kuntobille (कुण्टोबिल्ले) or Kuntalapi (कुण्टलपि).

In this game, a big rectangle is divided into 8 smaller rectangular regions as shown in Fig. 18. The player has to limp on one foot only throughout the game, and push with that foot a small disc or a flat piece of stone (of radius about 2 or 3 cm) from A to B along the path shown in the figure. He will be out if his foot,

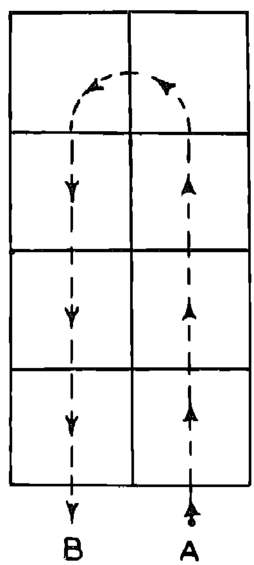


Fig. 18.

when resting on the ground, goes outside the big rectangle, or the disc does not lie entirely within one of the smaller rectangular regions or if it touches any boundary line. During the game, even if the disc or the foot falls very close to a boundary line, so long as it belongs entirely to a single region, we can draw a disc round it so that it also belongs entirely to the same region as the player's foot or the disc or piece of stone. Children test it by actually drawing a line between the foot mark and the boundary line that belongs to the same region as the foot mark. If such a line cannot be drawn, then the player is out.

Each of the small rectangles is a different region. The player has to limp across the boundary between them to go from one rectangle to another.

Example 5 :—

In the game of tennis or badminton, if the ball falls within the court, it is always possible to place a sufficiently small circle round the mark made by the ball on the ground so that this circle also belongs entirely to the same court.

In athletic games like throwing the disc or the javeline, the athlete's feet must be entirely within the circle marked in white. His feet should not even touch the white mark. This implies that there should always be some space, however narrow it may be, between his feet and the white mark.

Example 6 :—

Suppose a soldier is practising firing in a practising ground which is fenced all round. His maximum range must be less than his own distance from the nearest point of the fence, because his bullet must fall only within the ground. If he goes closer to the fence his range must be reduced. As he goes closer and closer to the fence, he will have to keep the gun closer and closer to himself, so that it may not touch the boundary. When he is very close to the fence, he may have space just enough to turn round with the gun.

Example 7 :—

Imagine you are standing in a cricket field inside the boundary line. You can hold an open umbrella right above your head so that at noon the shadow of the umbrella falls within the boundary. If you are far enough from the boundary, you can hold a big umbrella. But if you go nearer and nearer the boundary, you can still make the shadow fall w thin the boundary by reducing the size of your umbrella suitably.

That is, so long as you are within the boundary, however close you may be to it, you can still hold a sufficiently small umbrella over your head and manage to make the shadow fall entirely within the boundary. That is, every point within the boundary is an inner point.

Another way of stating that every point of a region R is an inner point, is that a region is also an open set of points. Why we say it is open is because it is open at every one of its points. That is, if P is a point of a region R, then a sufficiently small dot or disc round P, or a sufficiently small surrounding (or neighbourhood) around P also lies entirely within the region, as illustrated by the above examples. (Fig. 16 and 17). Therefore

A region is an open, connected set of points.

In the language of Modern Mathematics, the name for a region is *Open-Connect*.

1.4 Boundary Point, Boundary of a Region

In the examples studied above, (Ex. 2–7) if we proceed still further, we come to a point Q at which, however small the dot may be round the point Q, the dot contains a part belonging to R and also a part not belonging to R (Fig. 16, 17). That is, the dot, however small it may be, cannot belong entirely to R. Therefore, Q is not an inner point of R. Such a point is called a *boundary point* of R

Hence :

Q is a boundary point of R if every disc surrounding Q contains at least one point belonging to R and at least one point not belonging to R.

Boundary :

The set of all boundary points of a region is the boundary of the region.

Figure 16 shows that the line *l* of Example 2 is the common boundary of the two half-planes H_1 and H_2 into which *l* divides the plane. Figure 17 shows that the circle C of Example 3 is the common boundary of the regions R_1 and R_2 into which C divides the plane.

1.5 We will consider some more examples to illustrate the above properties.

Example 8 :—

Consider a circle with a segment AB drawn such that it belongs entirely to the inner disc region (Fig 19).

The remaining points of the plane form two regions, inner and outer.

The set of points on the circle is the boundary of the outer region. But the boundary of the inner region consists of the circle C and the segment AB. If P is any point on AB, any dot round P contains points of the inner region and a part of the segment AB which does not belong to the inner region.

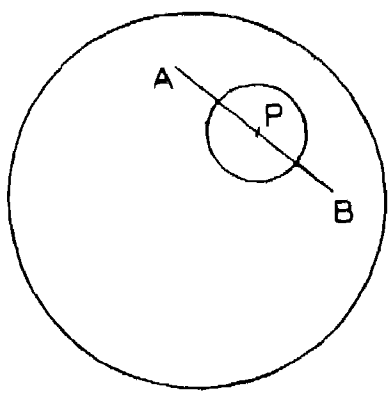


Fig. 19.

Example 9 :—

Let C_1 and C_2 be two disjoint circles as shown in Fig. 20. These divide the plane into the 3 regions —R_1, the inner disc of C_1, R_2, the inner disc of C_2, and R_3, the region outside C_1 and C_2.

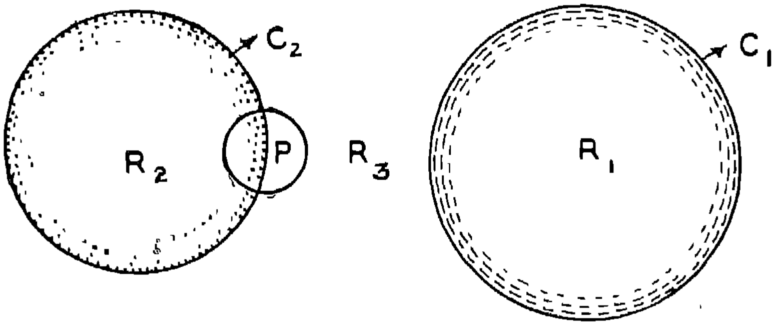


Fig. 20.

C_1 is the boundary of R_1, and C_2 is the boundary of R_2. Any disc round a point P on C_1 or C_2 contains points of R_3 and points not belonging to R_3 Therefore C_1 and C_2 together form the boundary of R_3.

Example 10 :—

Let C_1 be a circle inside the circle C_2. The two circles divide

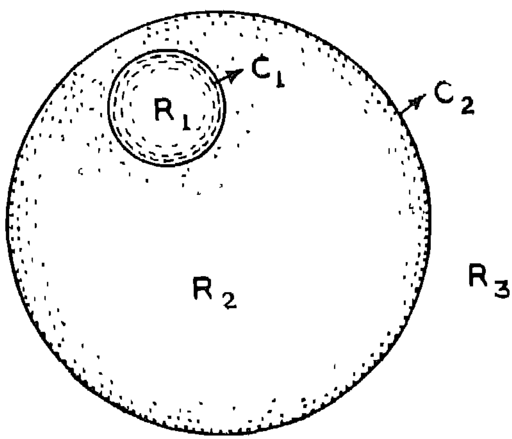


Fig. 21.

the plane into the three-regions R_1, R_2, R_3 as shown in Fig. 21.

The boundary of R_1 is C_1. The boundary of R_2 is C_1 and C_2. The boundary of R_3 is C_2.

Example 11 :—

Consider the set of all points of the plane except the points of a ray OX (Fig. 22).

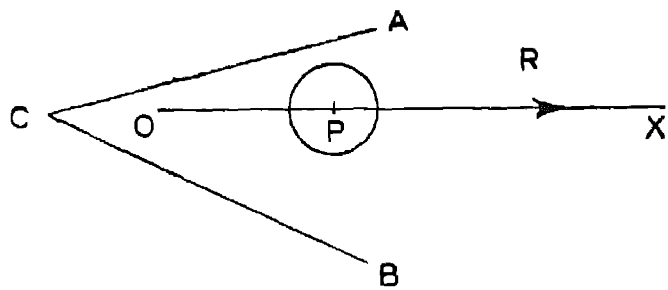


Fig. 22.

Let A and B be any two points of this set, that is, points of the plane not belonging to the ray OX. We can draw a segment chain ACB connecting them and belonging entirely to this set. That is, any two points of the set are connected. Therefore, a ray divides the plane into a single region R.

If P is any point on the ray OX, any disc round P contains points of R and a part of the ray not belonging to R. Thus, P is a boundary point and the ray is the boundary of the region.

Example 12 :—

If we remove only one point P from a plane, the remaining points form a single region R. Any disc round P contains many points of R and the point P which does not belong to R.

Therefore P is the only boundary point of R. There may be a figure which divides the plane into only one region having the figure as its boundary. Fig. 23 shows some such figures.

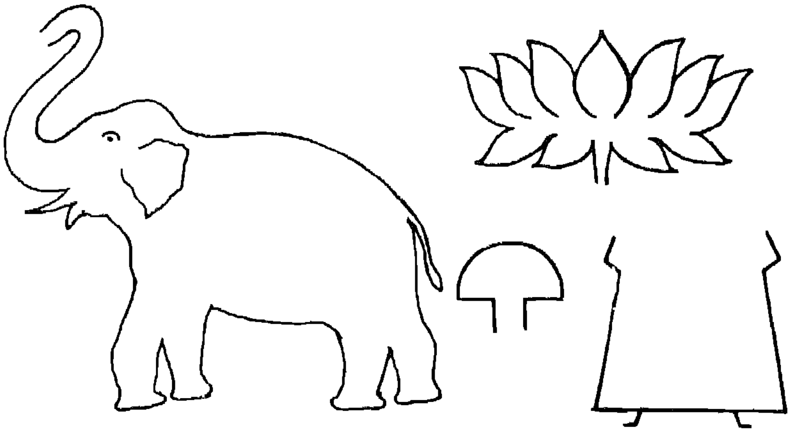


Fig. 23.

Example 13 :

Consider the set of all points of the plane except points on a line l. To this set add just one point P of the line l. This set R is connected, for if A and B are any two points on the opposite sides of

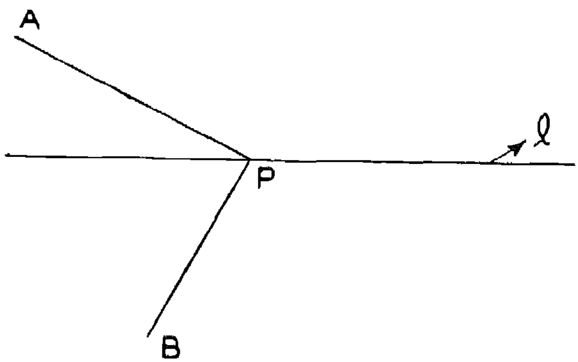


Fig. 24.

the line l (Fig. 24), it is always possible to draw the segment chain APB connecting them as shown in the figure, and belonging entirely to R, as P is a point of R.

But any dot round P contains many other points of l which do not belong to R. Therefore R contains a point P which is not an inner point. Therefore, though R is a connected set, it is not a region.

MAP OF MYSORE STATE

Fig. 25.

Example 14 :—

Figure 25 is a map of Mysore. It shows the division of the State into districts. Each of the 18 districts (except Tumkur) is a region. But Tumkur district consists of 2 separate regions (just as Pakistan) which are shaded in the figure. Tumkur town and Pavagada cannot be connected by a road which runs entirely within Tumkur District.

The curve AB consists of the common boundary points of Bangalore and Tumkur districts A is the common boundary point of the three regions, namely, Bangalore, Mandya and Tumkur districts. B is the common boundary point of Bangalore, Tumkur and Kolar districts.

The boundary of Hassan district has points common with the boundaries of Mysore, Mandya, Tumkur, Chitradurga, Chickmagalur, South Kanara and Coorg districts.

Example 15 :—

Fig. 26 (a) is the boundary of the single region into which it divides the plane. But (b) divides the plane into two regions. The whole curve is the boundary of the shaded region The circle C is the boundary of the outer region.

In (c), the whole curve is the boundary of the inner region, but the part of the curve from A to B is not the boundary of the outer region.

In (d), the inner disc region has only the circle C as its boundary whereas the whole curve is the boundary of the outer region.

In (e), we have three regions, R_1 the inner disc region, and R_2 and R_3, as shown in the figure. R_1 has only the circle as its boundary. R_2 has all the lines and the circle of the figure as its boundary. R_3 has the two rays AB and CD and the segment AC as its boundary.

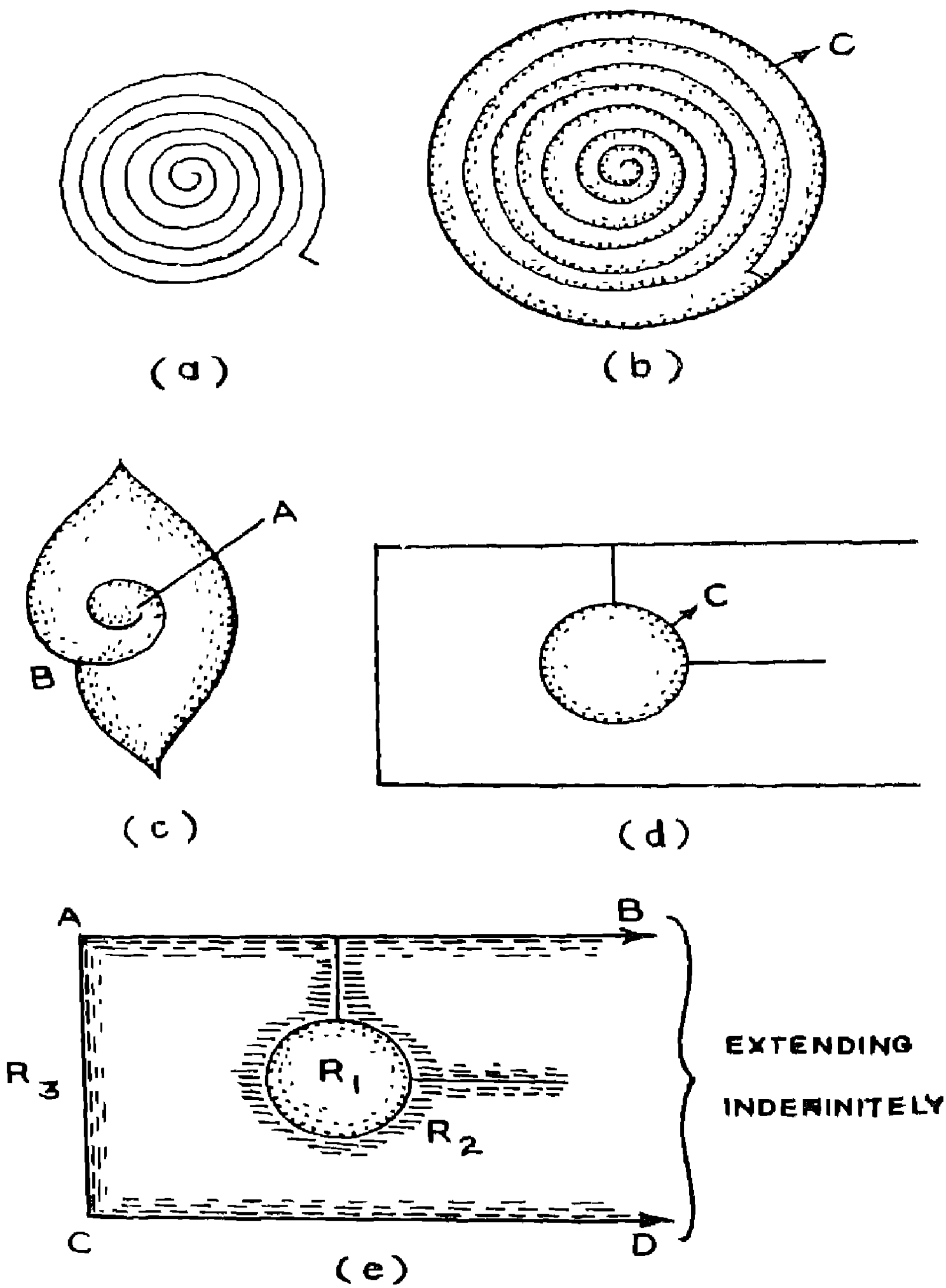


Fig. 26.

1.6 Axiom of Pasch

We now state an important property called the *Axiom of Pasch*, namely,

> If a line meets a side of a triangle (at an inner point, i.e., not at a vertex), then it intersects exactly one more side internally.

This is intuitively obvious. Study Figure 27.

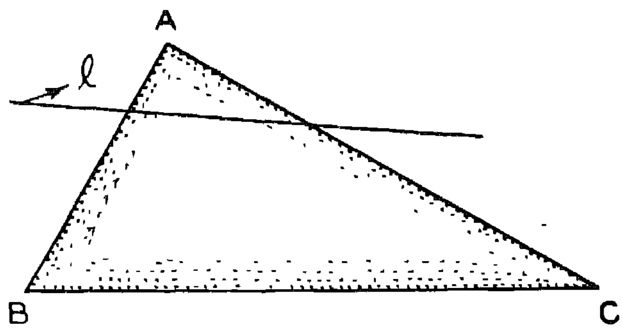


Fig. 27

Note. By repeatedly using this property defined by the Axiom of Pasch, one can logically deduce all the separation properties in a plane. It can be shown that the *Axiom of Pasch* and the *plane separation property* are equivalent, i. e., each of them can be proved by assuming the other to be true.

1.7 Jordan Curve

The easiest way to divide the plane into two regions in various ways is to draw simple (i.e., not self-crossing) closed continuous curves on the plane. Such a curve will divide the plane into exactly two regions Both these regions have the curve as their common boundary. Such curves are called *Jordan Curves* in honour of a reputed French mathematician, C. Jordan (1838–1922). It will be quite

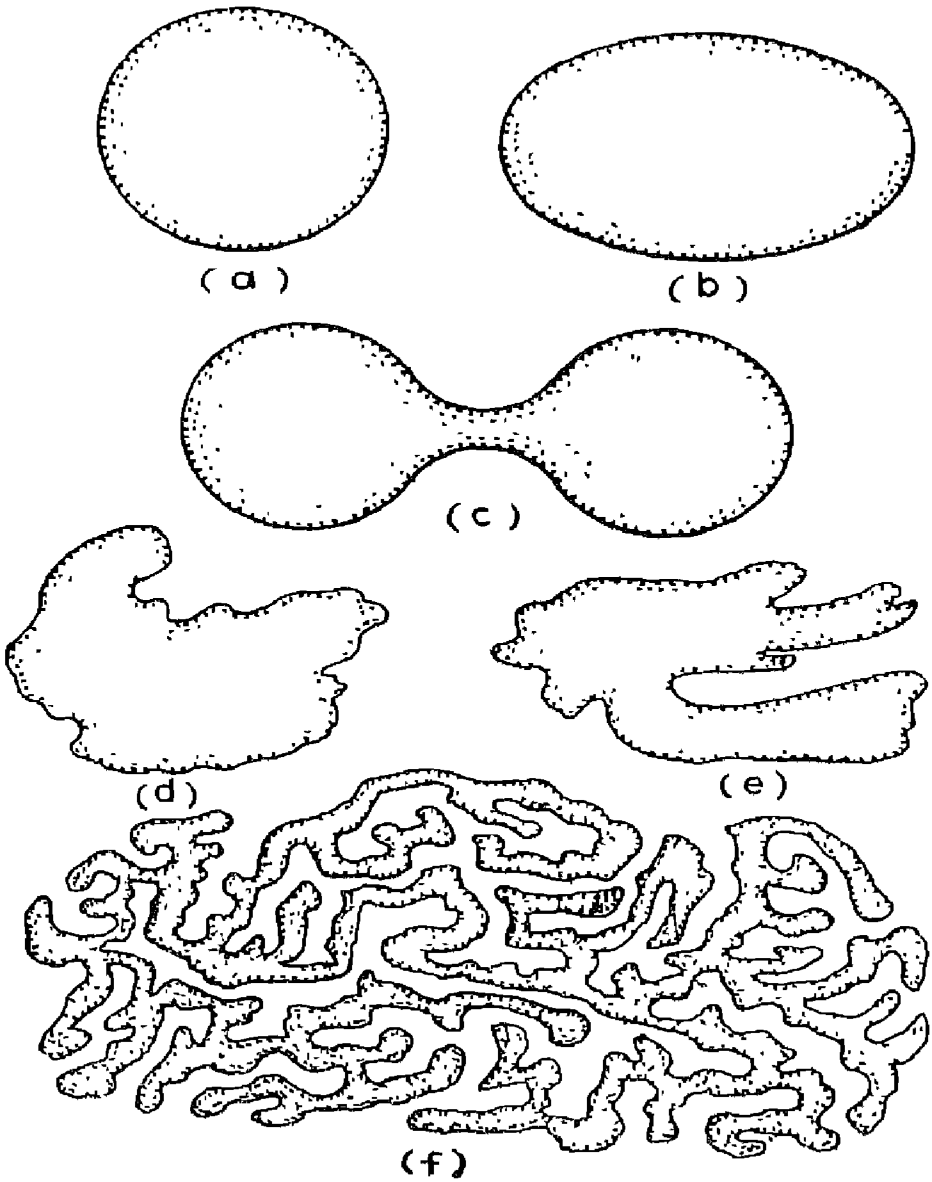


Fig. 28.

entertaining to start with the circle as the simple Jordan curve and draw more complicated Jordan curves. Some are shown in Figure 28. You can draw more complicated curves and shade the inner region, or take some points arbitrarily in the plane and find by trial and error whether they belong to the inner or the outer region.

Simple Polygon — A triangle is a Jordan Curve made up of 3 line segments [Fig. 29 (b)]. A simple quadrilateral is a Jordan Curve made up of 4 line segments [Fig. 29 (c)].

A simple polygon of n sides is a Jordan Curve made up of n line segments [Fig. 29]. That is, the only common point, *if any*, of any two segments is their common end point. These common end points are called the vertices. The segments are called the sides of the polygon. A pair of end points of any side will be called *adjacent vertices*. A line joining non-adjacent vertices is a *diagonal*. Some diagonals are shown in dotted lines in Fig. 29.

A triangle has no diagonal. Figure 29 (c) shows that both the diagonals of the quadrilateral lie in the inner region. They are called the *inner diagonals*. In Fig. 29 (g), *1* is an inner diagonal. But *2* belongs entirely to the outer region. It is called an *outer diagonal*. In Fig. 29 (d), the simple pentagon (five-sided polygon) has the two inner diagonals namely, *1* and *2*, and one outer diagonal, namely, *3*. The diagonals *4* and *5* cut the sides of the pentagon. A part of each of these belongs to the inner region and another part to the outer region. These are called *crossing diagonals*.

Example 16.

Let R_1 be the inner region and R_2 the outer region into which a simple polygon divides the plane. It is interesting to notice that if 2 points P and Q belong to the same region, any path consisting of connected line segments, connecting P and Q, either does not cross the polygon or it crosses the sides of the polygon at an even number of points, namely, at $C_1, C_2, \ldots\ldots, C_8$ (Fig. 30).

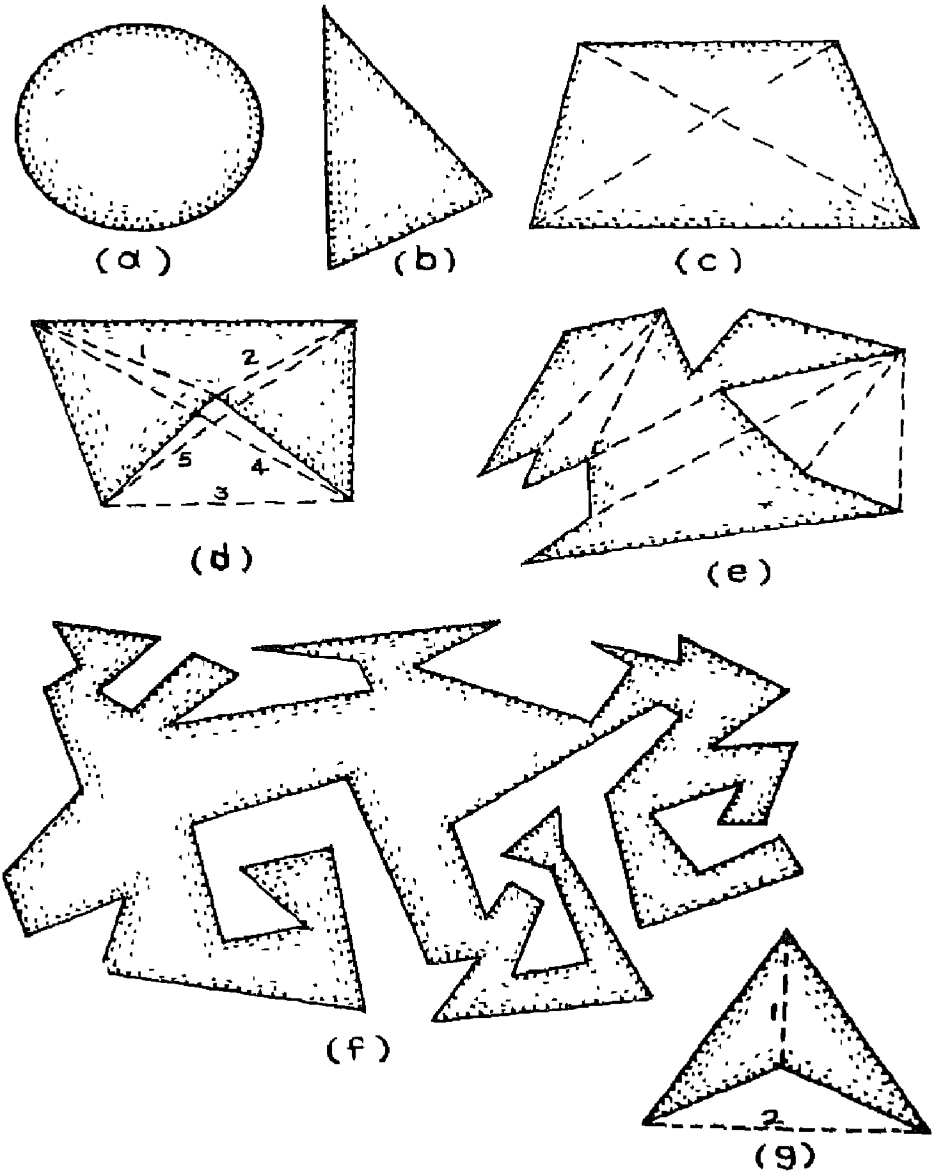


Fig. 29.

If P and S belong to different regions, one path connecting them has 3 crossings A_1, A_2, A_3, and the other 7 crossings B_1, B_2, B_3, ... B_7.

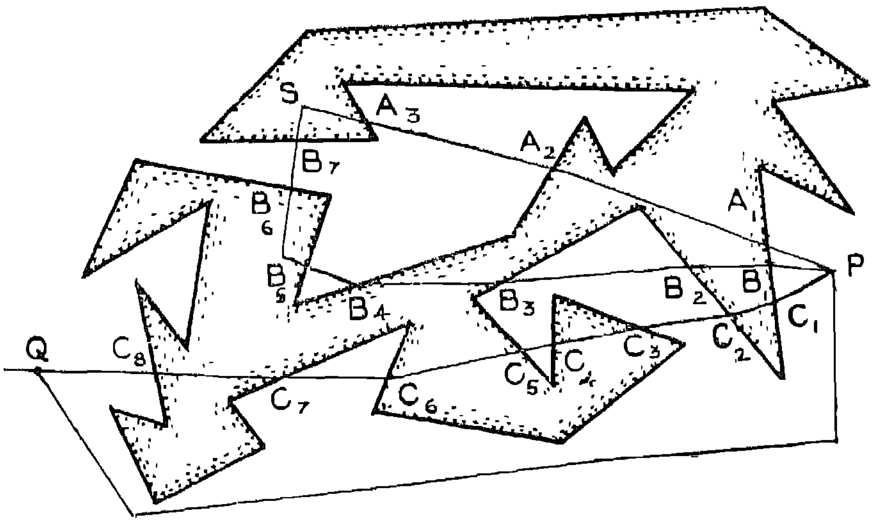
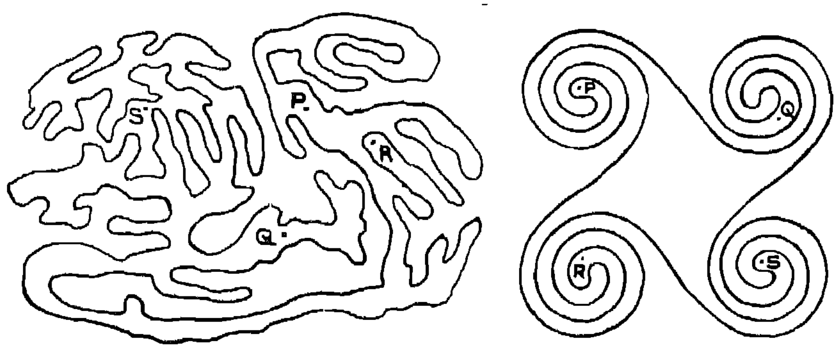


Fig. 30.

That is, such a path has always an odd number of crossings.

EXERCISE – 1.1

1. Into how many regions is the plane divided by (i) Figure. 31 (a) and (ii) Figure. 31 (b) ?



(a) (b)

Fig 31.

Which of the points P, Q, R, S marked in the figure belong to the same region and which to different regions? Draw segment chains connecting points belonging to the same region and running entirely in the region.

2. Fig. 32 divides the plane into a single region. Connect A and B by a segment chain belonging entirely to the region.

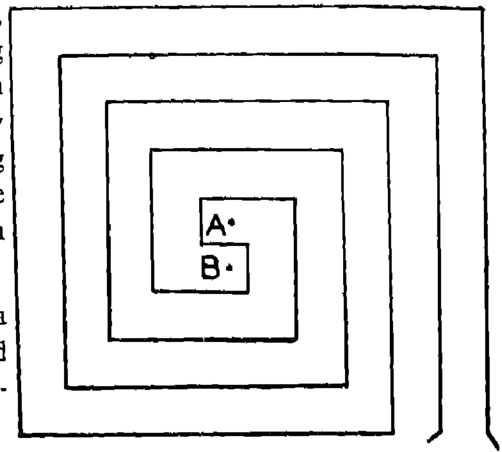


Fig. 32.

3. Into how many regions does each of the following figures (Fig. 33) divide the plane? Colour different regions differently. Classify the marked points.

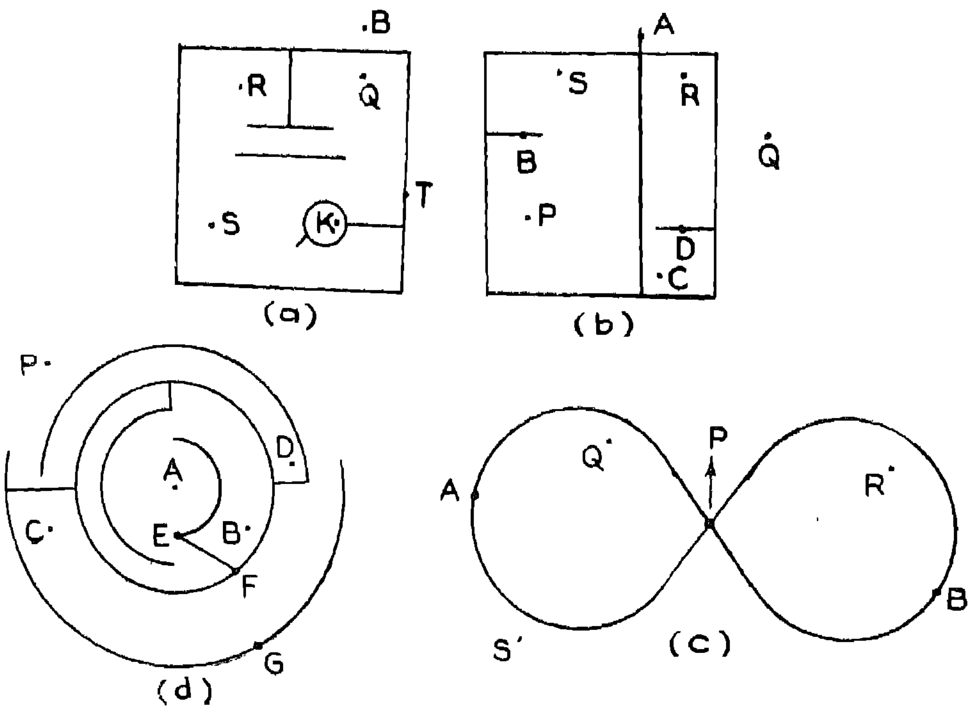


Fig. 33.

4. Examine the number of regions into which the figures (a), (b), (c), (d), (e) (Fig. 34) divide the plane. Colour them differently and name their boundaries.

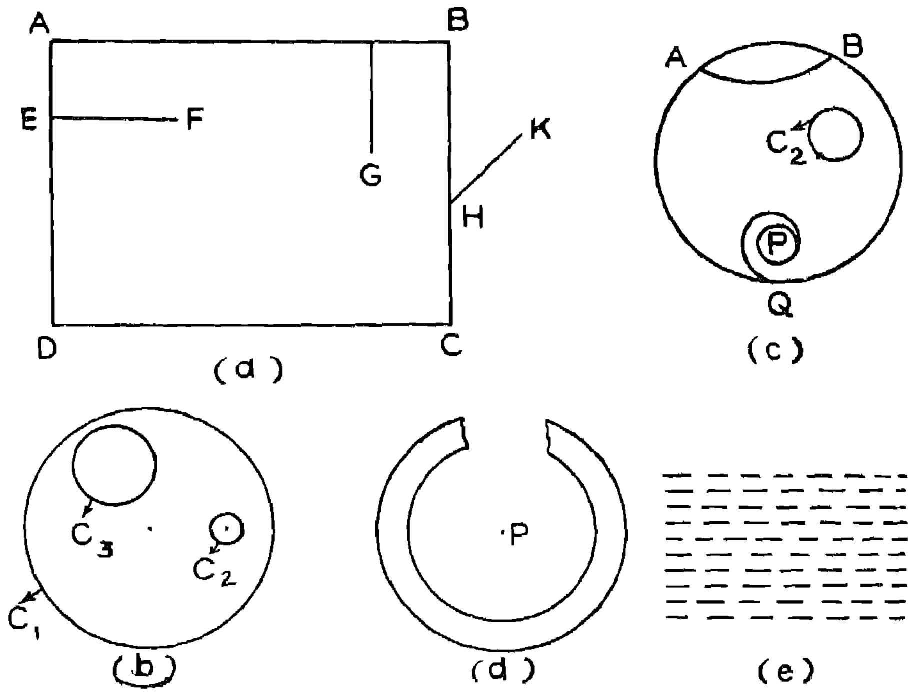


Fig. 34.

5. A circle is divided into four congruent arcs by the four points *1*, *2*, *3* and *4* (Fig. 35 a). Draw segments joining each of these to only one other point so that no two segments intersect. This can be done by joining *1*, *2* and *3*, *4*. Thus the inner disc region is divided into the 3 regions R_1, R_2 and R_3 Name their boundaries

We cannot join (*1*, *3*) and (*2*, *4*) because the segments so obtained intersect. By joining (*1*, *4*) and (*2*, *3*), [Fig. 35 (b)] we get the same pattern as the first. So we say that both these belong to the *same type*. Therefore, in this case we have only *one* type of such joins.

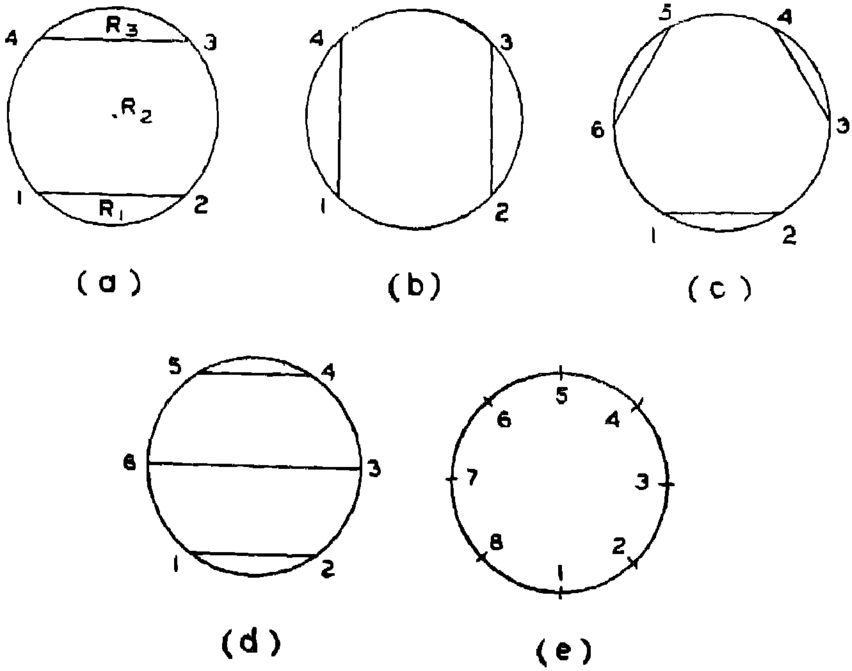


Fig. 35.

If we take six points *1, 2, 3, 4, 5, 6* dividing the circle into congruent arcs, verify in this case that we can have only two types, (c) and (d).

What is the number of regions into which the inner disc is divided in each case? Colour them differently and name their boundaries.

1, 2, 3, 4, 5, 6, 7 and *8* are the eight points dividing the circle into eight congruent arcs in Fig (e). In how many different *types* can you join them as above? Draw figures showing all possible types when 12 points are taken on the circle. In each case how many regions do you have? Colour them differently and name their boundaries

1:8 Convex Region

A convex region has the following additional property:—

> The segment joining *any* two points of a convex region belongs entirely to the region.

Example 17:—

A line divides the plane into two convex regions R_1 and R_2, such that if A and B belong to R_1, the segment AB also belongs to

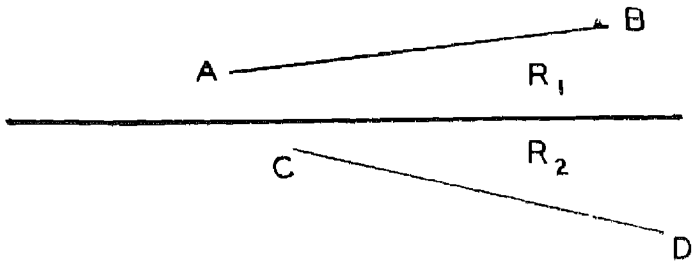


Fig. 36.

R_1, and if C and D belong to R_2, the segment CD also belongs to R_2

That is, A and B $\in R_1 \Rightarrow$ the segment AB $\in R_1$;
and C and D $\in R_2 \Rightarrow$ the segment CD $\in R_2$

Example 18·—

A circle divides the plane into two regions. Let A, B be any two points of R_1, the inner region. Then the segment AB $\in R_1$. Therefore the inner region of a circle is convex [Fig. 37.]

The outer region R_2 contains points like E, F such that the segment EF $\in R_2$. But it also contains points like C, D where the segment CD $\notin R_2$. Therefore R_2 is not convex.

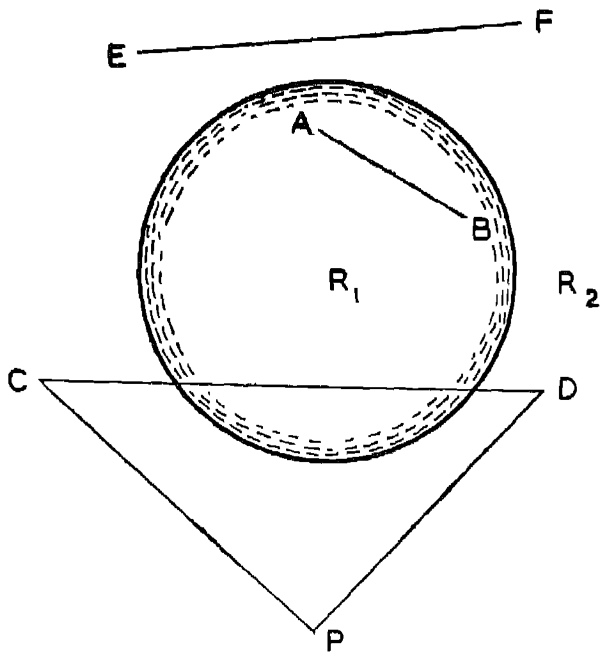


Fig. 37.

Example 19 :—

Similarly the inner region of a triangle is convex The outer region is not convex. Give reasons (Fig. 38).

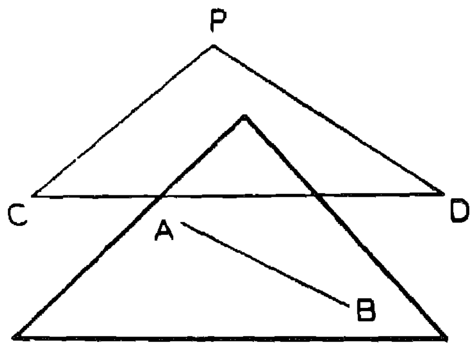


Fig. 38.

Example 20 :—

Two intersecting lines divide the plane into exactly four regions R_1, R_2, R_3, and R_4. All these are convex [Fig. 39]. If A, B are two

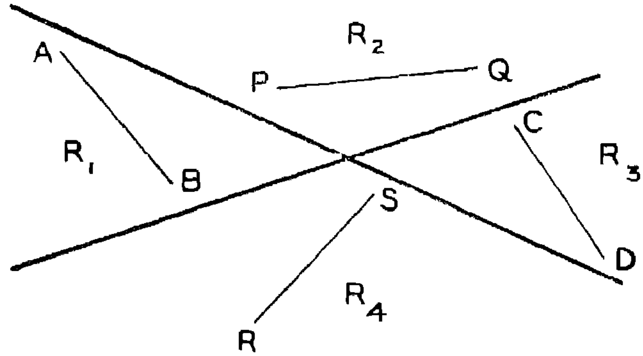


Fig. 39.

points of R_1, then the segment $AB \in R_1$

Therefore R_1 is convex. Similarly, R_2, R_3, R_4 are convex.

Example 21 :—

Two parallel lines divide the plane into three convex regions: R_1, R_2, R_3 (Fig. 40).

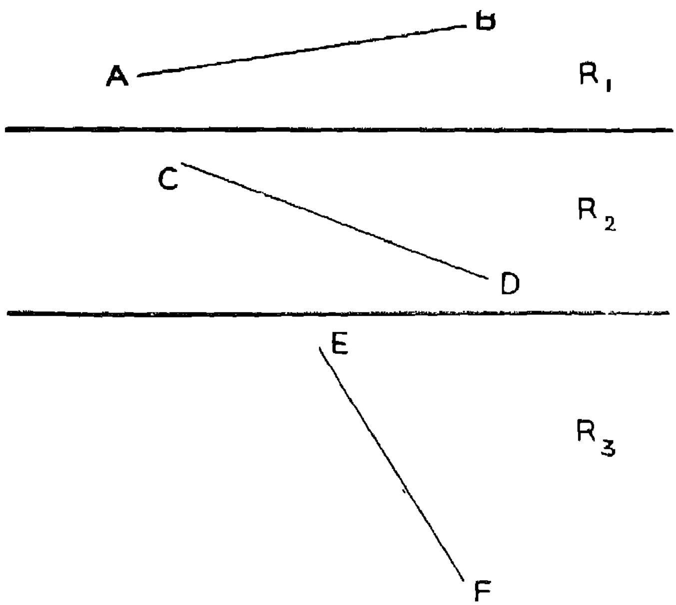


Fig. 40.

Example 22 :—

Verify the following statements in the figures. The whole plane is one region in general.

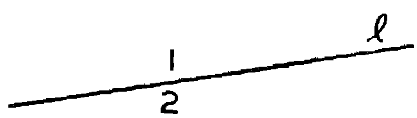


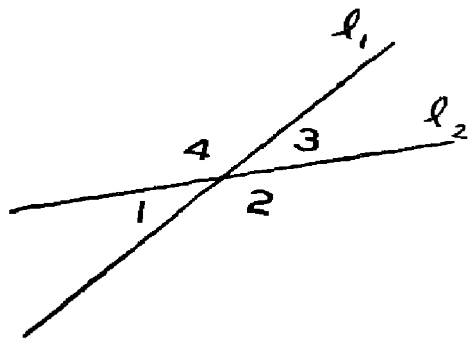
Fig. 41.

1 line divides the plane into

$1+1 =$ 2 convex regions. [Fig. 41].

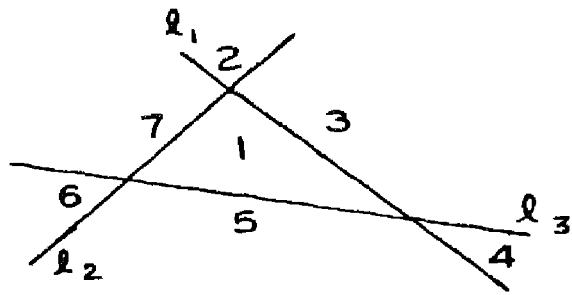
2 lines divide the plane into
$1+1+2 =$ 4 convex regions. [Fig. 42].

Fig. 42.



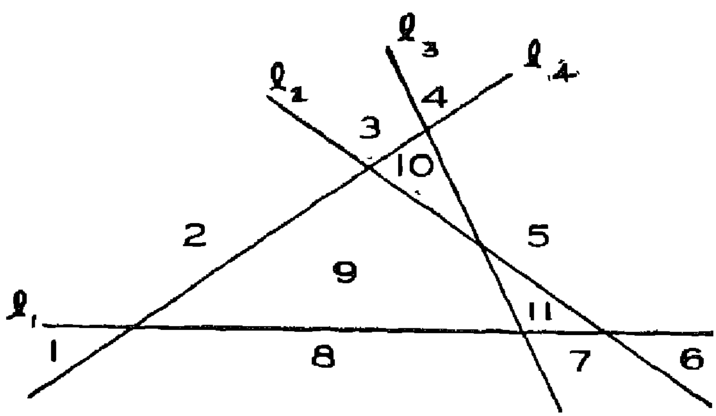
3 lines divide the plane into
$1+1+2+3 =$ 7 convex regions. [Fig. 43].

Fig. 43.



4 lines divide the plane into
$1+1+2+3+4 =$ 11 convex regions. [Fig. 44].

Fig. 44.



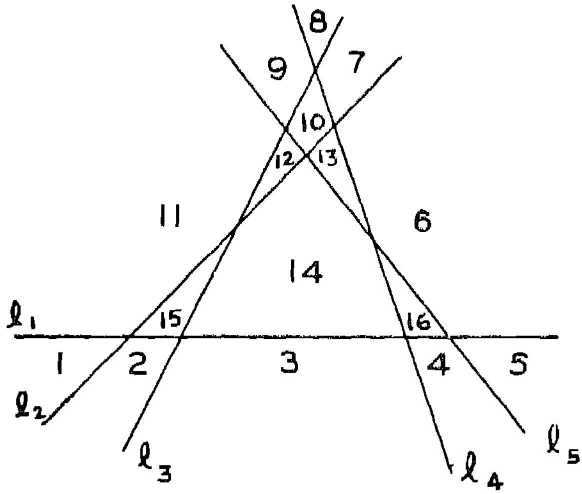


Fig. 45.

5 lines divide the plane into

$1+1+2+3+4+5 = 16$ convex regions (Fig. 45), and so on.

Now you know how to write down the number of convex regions into which 6 lines, 7 lines etc., divide the plane.

(All the lines must be drawn in general position i.e., no two lines are parallel. No three lines meet in a point.)

Into how many regions do 10 lines in general position divide the plane?

Example 23 :—

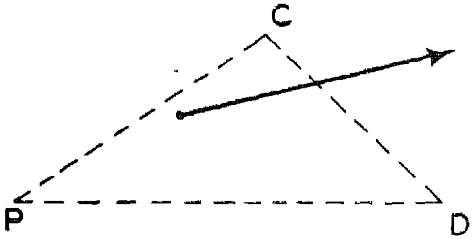


Fig. 46.

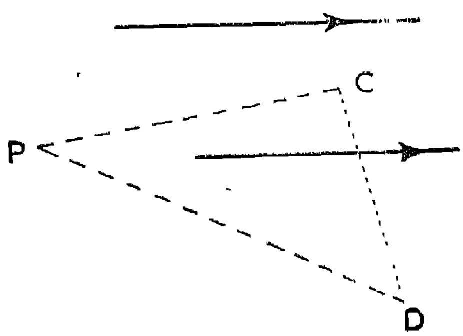


Fig. 47.

A ray or a segment divides the plane into only one region. Fig. 46 shows that it is not convex. Two rays which have no common point divide the plane into only one region (Fig. 47), which is not convex.

Two rays which have a common point divide the plane into two regions. Only one of these is convex (Fig. 48), unless the two rays belong to the same line. Explain why ?

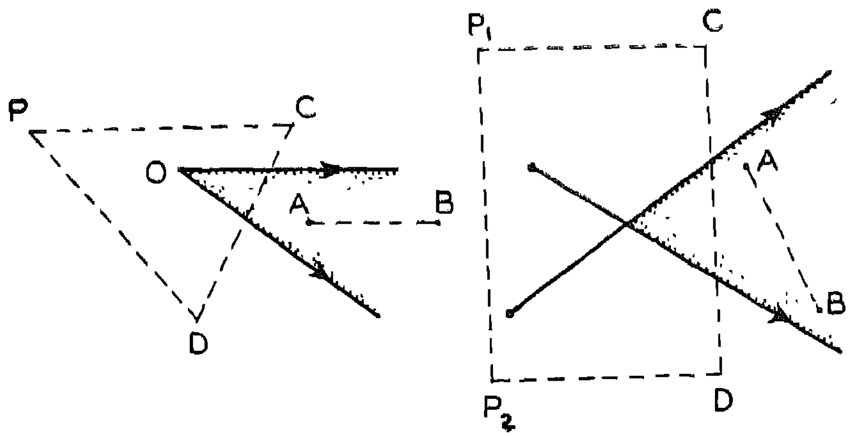


Fig. 48.

Example 24 :—

Each of the following figures (Fig. 49) divides the plane into two regions. The inner region is convex. The outer region is not convex. A, B are points of the convex inner region. Therefore the segment A B belongs entirely to the inner region. There are points like C and D both belonging to the outer region. They are connected by the path C P D belonging entirely to this region. But the segment does not belong entirely to the outer region. This shows that the outer region is not convex.

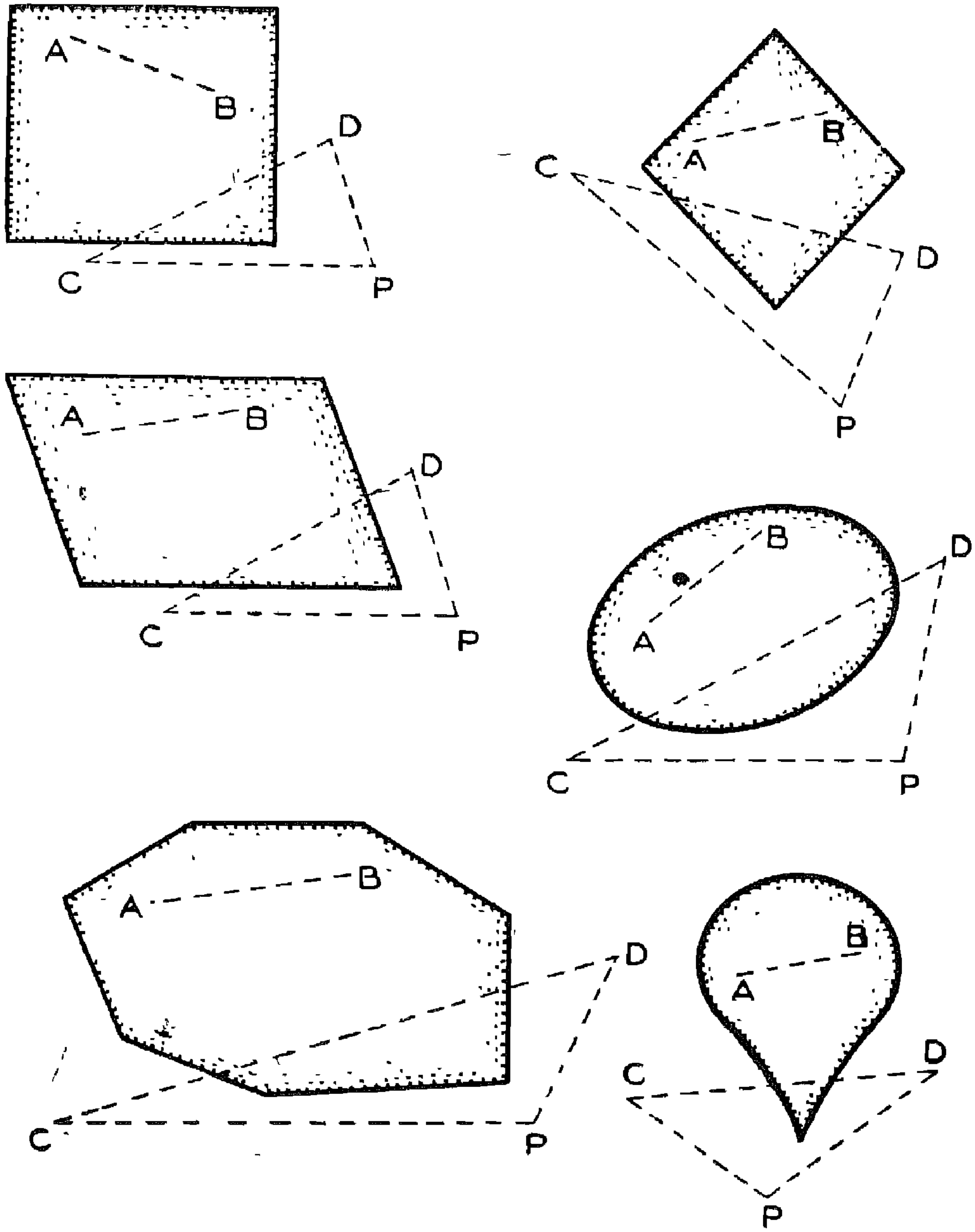
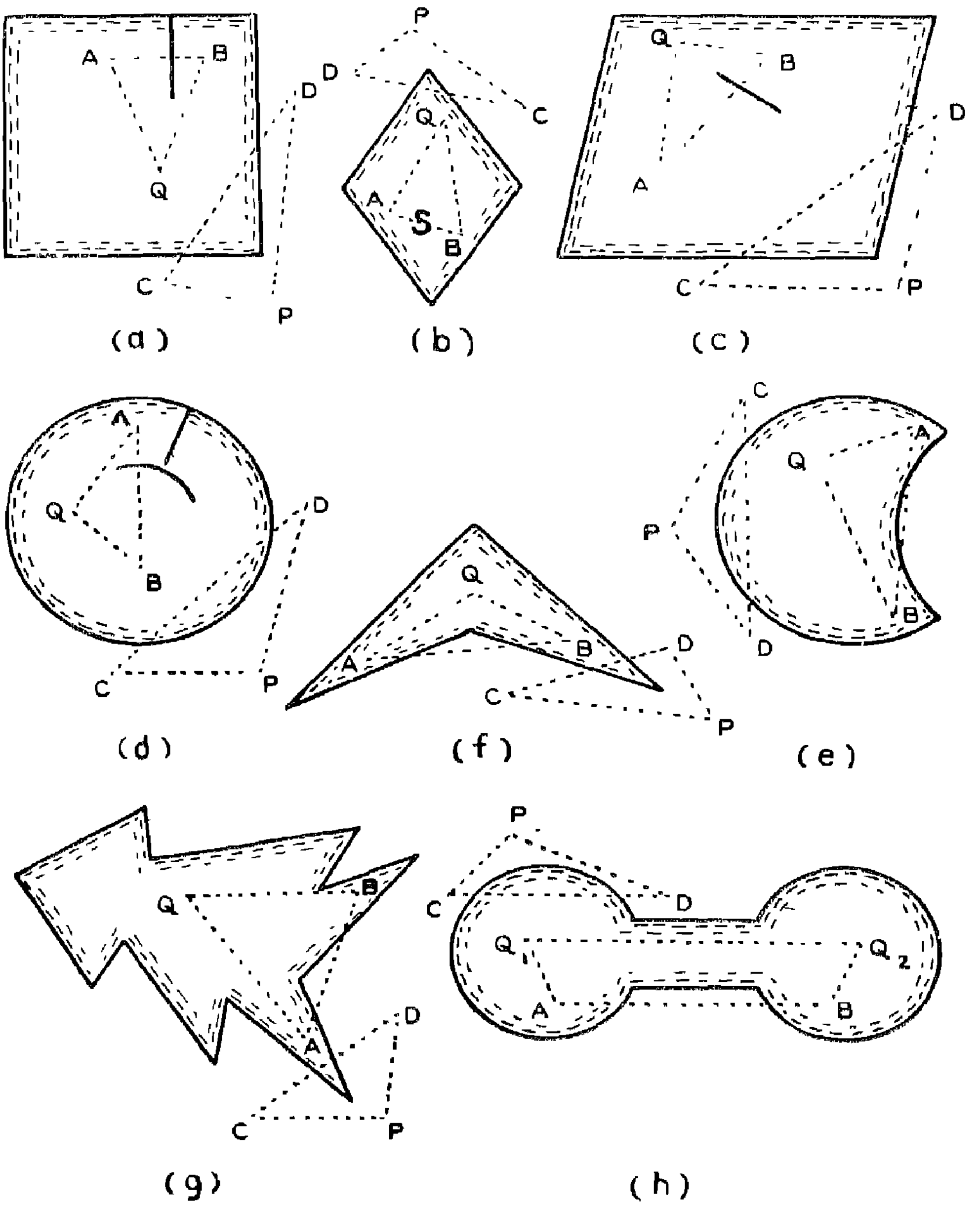


Fig. 49.



A
B
Q
C
D
P
(a)
P
D
C
Q
A
S
B
(b)
Q
B
A
D
C
P
(c)
A
Q
B
D
C
P
(d)
Q
A
B
D
C
P
(f)
C
A
Q
P
B
D
(e)
Q
B
A
D
C
P
(g)
P
C
D
Q_1
Q_2
A
B
(h)

Fig. 50.

Example 25 :—

Each of the figures on page 36 divides the plane into two regions which are not convex. Consider points like A and B belonging to the inner region. They are connected by the path A Q B or A Q_1 Q_2 B belonging entirely to the inner region. But the segment AB contains a point or points of the boundary of the region. Therefore the inner region is not convex (Fig. 50).

There are points like C and D belonging to the outer region. They are connected by the path C P D. But the segment C D cuts the boundary. Therefore the outer region also is not convex.

1.9 An Important Deduction

We will deduce that

> The common part of two convex regions is again a convex region.

The common part of two sets R_1 and R_2 is called their intersection and is denoted by the symbol $R_1 \cap R_2$, i.e., a point belongs to $R_1 \cap R_2$ if it belongs to both R_1 and R_2.

Let R_3 be their common part.

Then we have to deduce that R_3 also is convex.

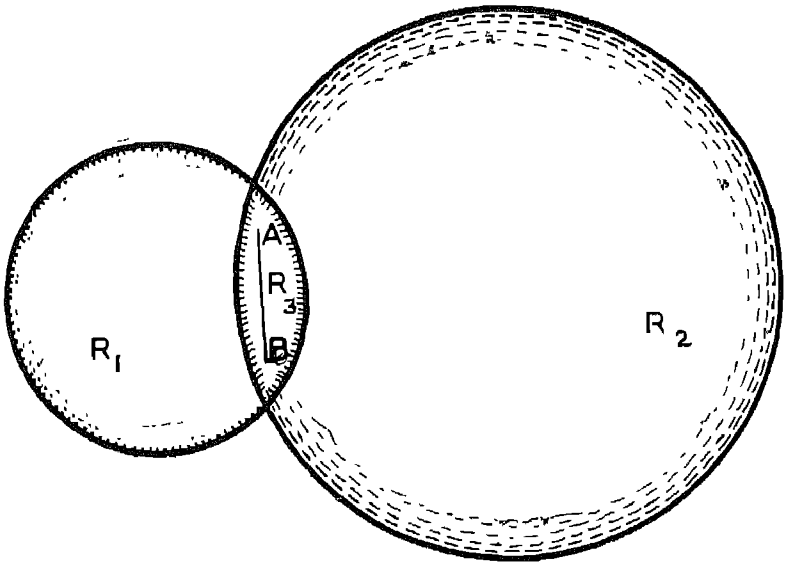


Fig. 51.

Let A,B be any two points of R_3 (Fig. 51).

Every point of $R_3 \in R_1 \Rightarrow A,B \in R_1$.

And R_1 is convex $\Rightarrow$ the segment $AB \in R_1$ ..... (1)

Similarly, every point of $R_3 \in R_2 \Rightarrow A,B \in R_2$.

And R_2 is convex $\Rightarrow$ the segment $AB \in R_2$ ......(2)

(1) and (2) $\Rightarrow$ the segment $AB \in R_1$ and also to R_2.

Therefore it belongs to their common part R_3.

That is, $A,B \in R_3 \Rightarrow$ the segment $AB \in R_3$.

That is, R_3 is convex

Therefore the common part of two convex regions is again a convex region.

EXERCISE 1.2

1. Let three non-collinear rays start from O. They divide the plane into 3 regions. Similarly 4 rays from O divide the plane into 4 regions, 10 rays from O into 10 regions What is the number

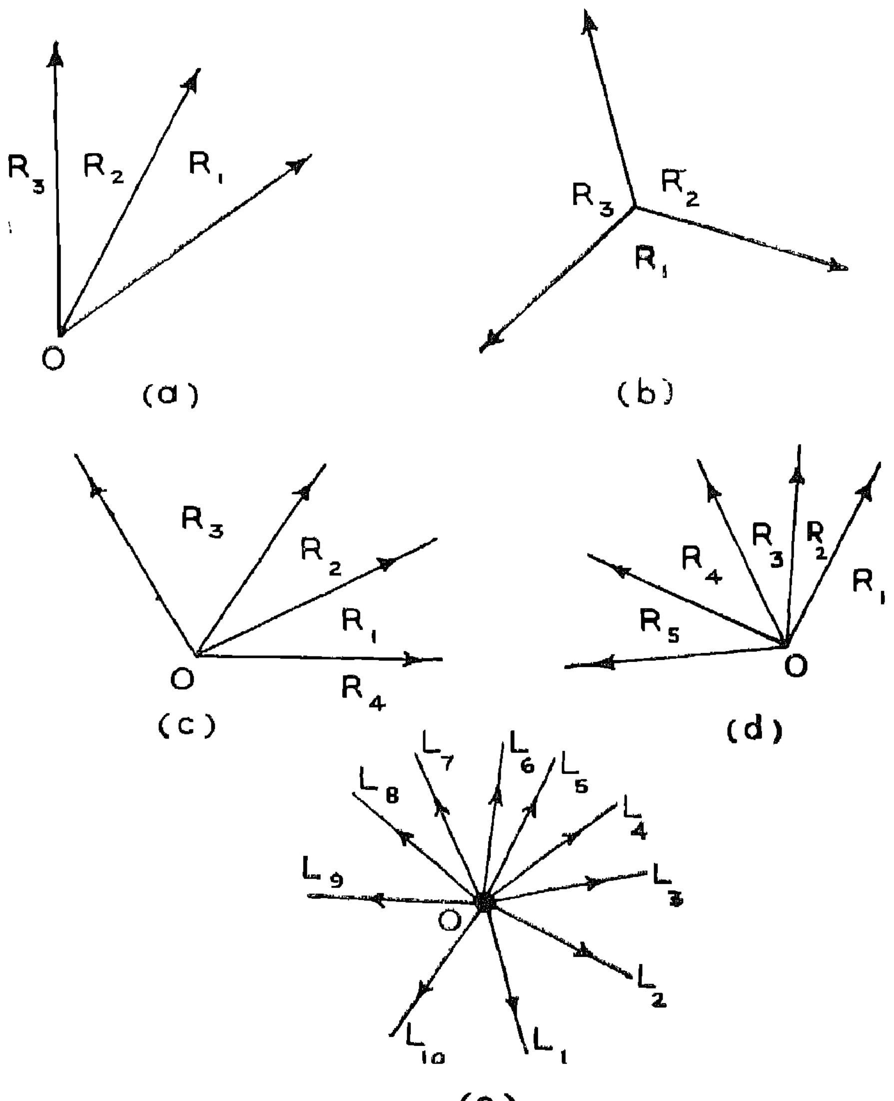


Fig. 52.

of *convex* regions into which a plane is divided in each of the cases shown in Fig. 52? Name them.

2. Examine the following figures and classify them into convex regions and regions that are not convex.

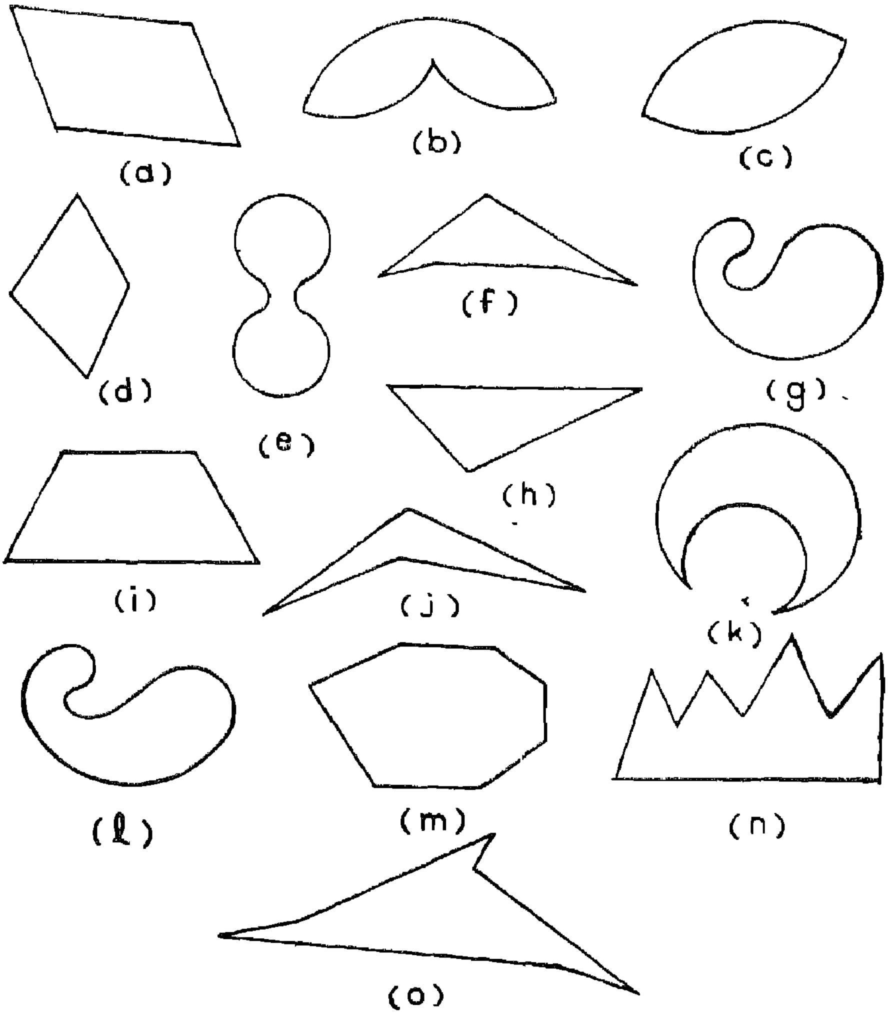


Fig. 53.

3. Choose a point inside a convex region. How many times will a ray through it cut the boundary of the region?

4. Which of the regions indicated by Roman numerals in Fig. 54 are convex ?

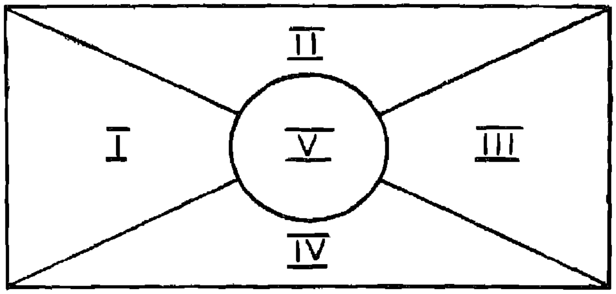


Fig. 54.

5. Is the common part of the inner regions of two intersecting circles a convex region? Explain by drawing figures.

6. Draw a plane quadrilateral (a figure with four sides) whose interior is convex.
Also draw a quadrilateral whose interior is not convex.

7. Can two lines in a plane separate the plane into two regions, three regions, four regions, five regions? Draw the lines in each case and specify the boundaries of the regions formed.

8. Draw two circles with centres A and B and radii equal to 1 cm and 2 cm where

(1) $|AB| = 4.0$ cm
(2) $|AB| = 0.5$ cm
(3) $|AB| = 2.5$ cm
(4) $|AB| = 3.0$ cm
(5) $|AB| = 1.0$ cm.

Into how many regions is the plane divided by the two circles in each case ? Colour different regions differently. Point out the convex regions among them.

9. Two circles have centres A, B and radii 3 cm and 5 cm. What should be $|AB|$ if:

(1) each circle lies outside the other;
(2) one circle lies within the other;

(3) they intersect in two distinct points;
(4) they touch externally;
(5) they touch internally?

10. Can three lines in a plane ever separate the plane into three regions, six regions, seven regions? Draw the figure in each case and specify the boundaries.

11. If one point is removed from a plane, then is the region so formed a convex region?

1.10 Regions in Space

Just as a line divides the plane into two convex regions, a plane divides space into two convex regions. One cannot go from one room to the next if there is no door in the wall between the rooms,

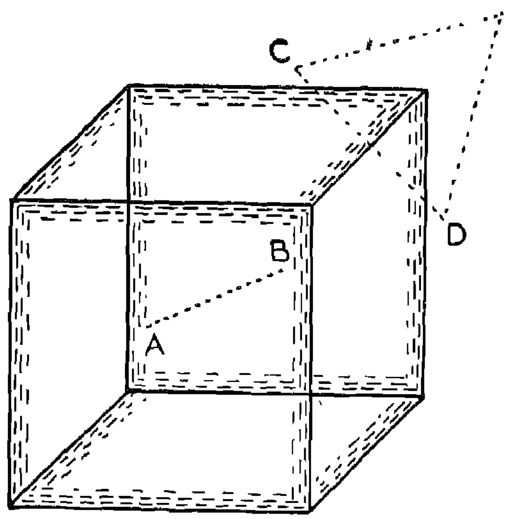


Fig. 55.

i.e., one has to cross the wall through a door. A rectangular box or a cube divides space into a convex inner region and the outer region which is not convex. Let A, B $\in R_1$, the inner region (Fig. 55). Then

the segment AB $\in R_1$. We can take two points C, D outside the box such that the segment CD cuts through the box. Therefore the outer region is not convex. One cannot touch anything inside a closed box from the outside. If you keep the only key of the lock inside a box and lock it by pressing the lock, you cannot take the key without breaking the box.

The inside of a ball is a convex region; the outside is also a region but it is not convex (Fig.56).

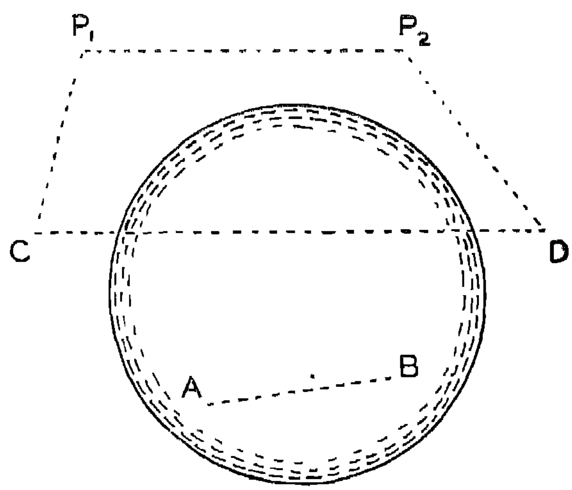


Fig. 56.

The skin of a tomato is the common boundary of the inner convex region and the outer region which is not convex.

The rubber tube of a bicycle wheel (or a tenniquoit ring) divides space into two regions and these are not convex. If A and B are two

points belonging to the inner region (Fig. 57), the segment AB does not always belong entirely to that region.

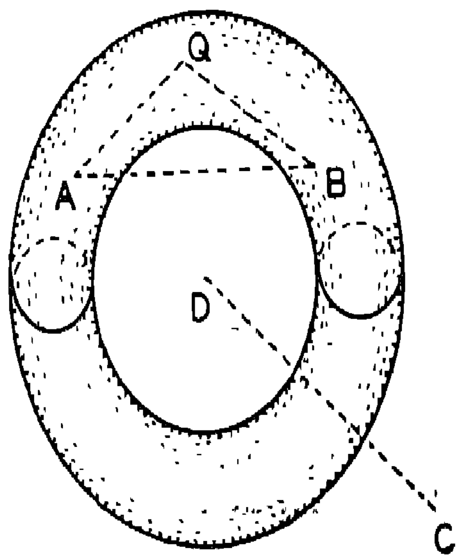


Fig. 57.

The outer region also is not convex. The segment joining two points like C, D of the outer region cuts into the inner region. This object resembles a thick bangle. This is also called an *anchor ring* or a *tore*.

Exercise :—

Give some examples of regions in space, which are not convex.

EXERCISE 1.3

1. What is the number of regions into which the plane is divided by each of the figures in the charts? Which of these regions are convex? How are the points represented by black dots and the numbers which are circled, related to these regions?

2. In each figure, consider any one point represented by a black dot or a number which is circled How many different regions have this point as a boundary point? What is the nature of each one of these regions?

3. Count the number of triangles in each one of these figures. How many of these are isosceles, and how many equilateral?

4. Count the number of quadrilaterals formed in each of these figures Classify them.

5. Which of the figures contain (a) one regular pentagon,
   (b) two regular pentagons?

6. Name the figures in which each region has as its boundary
   (a) a triangle, (b) a quadrilateral, (c) a pentagon.

7. Consider only the blue coloured segments. In each of these figures what is the number of regions into which they divide the plane? Which of these regions is convex?

8. Suppose the yellow segments are removed from these figures.
   (a) Do the remaining segments form a connected set?
   (b) How many different regions do you get?
   (c) How many of these are convex?
   (d) Show the boundary of each region.

9. Consider the set of (a) red *and* yellow segments only,
   (b) blue *and* green segments only,
   (c) red *and* green segments only,
   (d) yellow *and* green segments only, in each figure of the chart. Which of the above sets is connected? What is the number of regions into which the plane is divided by this? Which of these are convex? Show their boundaries.

10. Name the figures in which

    (a) no two segments of the same colour are parallel;
    (b) no two segments of the same colour are perpendicular;
    (c) no two segments of the same colour are either parallel or perpendicular;
    (d) any two segments of the same colour are either parallel or perpendicular.

11. In each of these figures (ignoring the colour differences) show, whenever it exists, the set of mutually congruent

    (a) segments,
    (b) triangles,
    (c) quadrilaterals,
    (d) pentagons,
    (e) hexagons.

12. Point out (a) four convex hexagons;
    (b) six convex heptagons (seven-sided polygons);
    (c) four convex octagons (eight-sided polygons) formed by some suitably chosen segments from Figure 21 of the chart.

13. In each figure show the symmetry lines (if any) of the set of

    (a) red segments,
    (b) yellow segments,
    (c) blue segments,
    (d) green segments,
    (e) purple segments,
    (f) red *and* yellow segments taken together,
    (g) red *and* blue segments taken together,
    (h) yellow *and* green segments taken together.

14. If the colours are ignored, what are the symmetry lines of each figure?

15. Which of these figures possess a line such that a reflection in the line maps all red segments on to green and all green segments on to red?

2

Angle

2.1 Orientation at a Point in a Plane - Orientation in a Plane

Consider a ray OA from O. Shade one side of it as shown in Figure 58. We adopt the convention that it is to be crossed from the unshaded side to the shaded side of it. We will now illustrate that the way in which this one ray from O is shaded determines how any other ray from O is to be shaded.

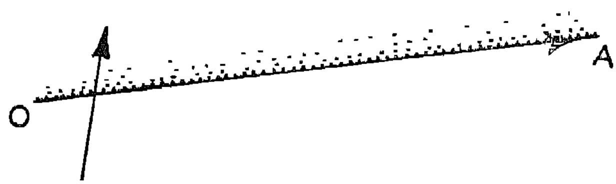


Fig. 58.

A pencil of rays divides the plane into regions We know that the number of these regions is same as the number of rays in the pencil.

Let us take 4 rays, OA, OB, OC, OD (Figure 59). They divide the plane into 4 regions. Call them R_1, R_2, R_3, R_4. We will set a (1, 1) correspondence between them. That is, a ray must correspond to only one region and a region must correspond to only one ray. This can be done in one of the following two ways. OA is the boundary of both R_1 and R_4. Therefore OA can correspond either to R_1 or to R_4.

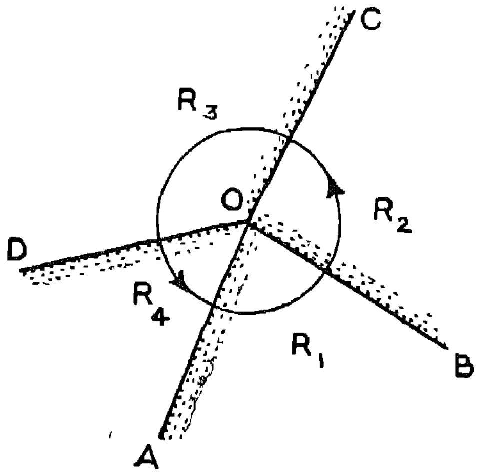


Fig. 59.

1) First let OA correspond to R_1. Shade OA on the side of R_1.
Then OB *must* correspond to R_2. Shade OB on the side of R_2.
OC *must* correspond to R_3. Shade OC on the side of R_3.
And OD *must* correspond to R_4. Shade OD on the side of R_4.

Draw a circle with O as centre cutting the rays at A, B, C and D. Now mark a direction along the circle as shown in Figure 59. Then as one proceeds round the circle in this direction one crosses OA at A from the unshaded side to the shaded side of OA. Observe that each of OB, OC, OD, is also crossed from the unshaded side to the shaded side of the same ray.

Thus it gives a definite direction or sense, namely, from A to B, from B to C, from C to D and from D to A of going round any circle with a point O of the plane as centre.

2) *Or* we could also have made OA correspond to R_4 and shaded OA on the side of R_4 (Figure 60).

Then OB must correspond to R_1. Shade OB on the side of R_1.
OC must correspond to R_2. Shade OC on the side of R_2.
And OD must correspond to R_3. Shade OD on the side of R_3.

Now mark a direction on the circle as shown in Fig. 60, opposite to that in Fig. 59. Going along the circle in this direction, one crosses every ray again from the unshaded side to the shaded side of the same ray. Therefore this also gives a definite direction or sense of describing the circle, namely, from A to D, from D to C, from C to B, from B to A.

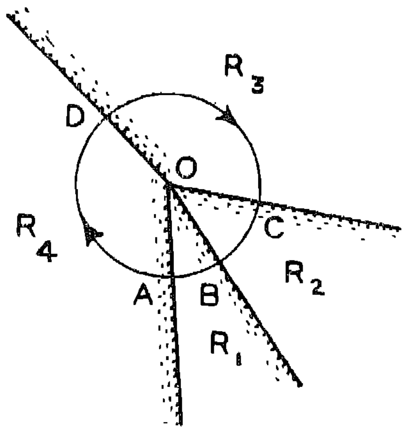


Fig. 60.

Thus we have two opposite senses of going round the same circle. In the first case, the rays are to be ordered as OA, OB, OC, OD. In the second case, they are ordered in the reverse way, namely, OD, OC, OB, OA.

Thus we have the two opposite ways of ordering a pencil of rays through any point O of the plane. These are called the two *orientations* at the point O of the plane. They are said to be opposite to each other.

You will see that the second of these is the direction in which the hands of a clock move and is called *clockwise*. It is called प्रदक्षिणे (*Pradakshine*) in our language, that is, as one proceeds along the circle the centre will always be to one's right. The opposite orientation is called *anti-clockwise*. It is अप्रदक्षिणे (*Apradakshine*).

[This motion round an object in the *pradakshina* sense is considered auspicious in all Hindu rituals.

We all know that before entering a temple, it is the Hindu custom to perform *pradakshina*, that is, to go round the temple in the clockwise direction. During marriage, the couple goes round the sacred fire in the same sense].

You have learnt orientation at a point and the way of shading the rays through that point corresponding to the orientation adopted. Now the orientation at any other point in the plane can be determined according to the following rule.

Suppose we have determined the orientation at the point *1* and want to determine the orientation at another point say *12* (Fig. 61).

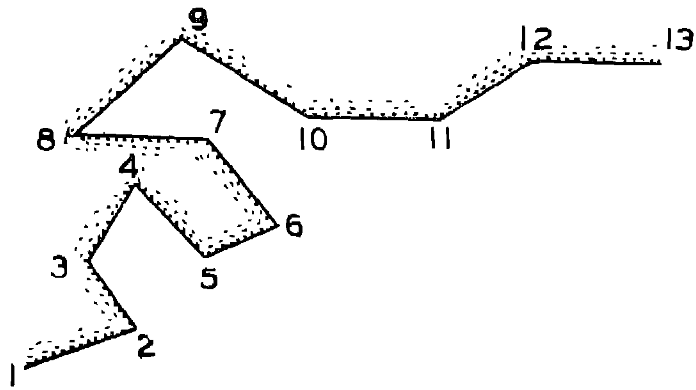


Fig. 61

Take a directed segment chain *1 - 2 - 3 .... -13* shown in the figure. We proceed to mark the orientation at the points *2, 3, 4, ..., 12* step by step. Suppose the orientation at the point *1*, determines the shading of the ray *1 - 2* as in Fig. 61 ; that is, we have shaded that side of the ray *1 - 2* which *contains* the point *3;* therefore we shade that side of the ray *2 - 3* which *contains* the point 1. This determines the corresponding orientation at the point *2.* Now observe that the shading of the ray *2 - 3* is on that side of it which *does not contain* the point

4; therefore we shade the ray *3 - 4* on that side of it which *does not contain* the preceding point *2*, we proceed in this manner and determine the orientation at the point *12*. It can be shown using the *Plane - Separation - Axiom* that we obtain the same orientation at the point *12* whatever be the directed segment chain by which we join the point *1* to the point *12* Thus:

> The orientation obtained at the same point by following different segment chains will be the same.

This is a basic property of the plane. There are surfaces on which this property does not hold.

Observe that in writing the figure 8 in the usual way, the upper and lower circles are described in opposite senses as shown in Figure 62.

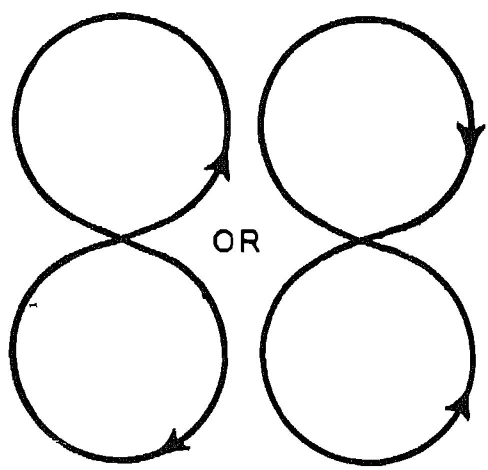


Fig. 62.

2.2 Effect of Reflection on Orientation

Every day you will be looking at your reflection in a mirror. Therein you find that your left and right sides are interchanged. Or your right-handed sister appears to be left-handed while combing her hair, when seen through a mirror. For this reason, such a reflection is called *lateral reflection*.

But have you looked at your double reflection? That is, take another mirror and through it look at your image in the first mirror. Now you see that your left and right sides appear in their proper places, that is, there is no lateral reflection. If you can manage to see your image after 3 reflections in 3 mirrors, you will find the same old lateral reflection again. What is the reason?

We shall analyse this problem as follows :—

Let ABC be a circle round O (Fig. 63). Let the orientation ABC be clockwise. Reflect it in a line l Let l denote reflection in l l (A), l (B), l (C), is a path in the anticlockwise direction along the circle round l (O). That is, a reflection in a line changes the orientation

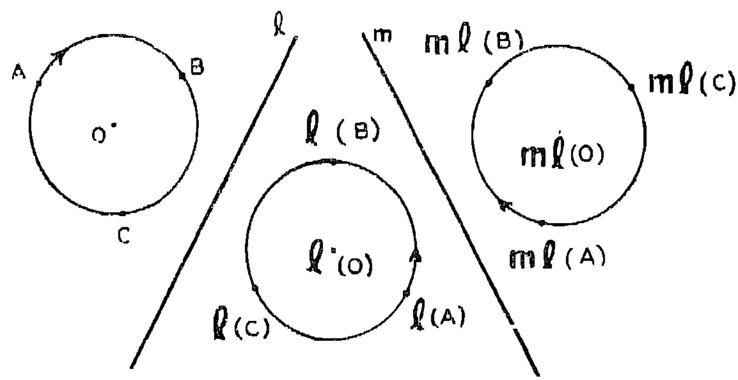


Fig. 63.

of the plane to its opposite one. You can see in Fig. 63 that another reflection in another line m changes the orientation again to the clockwise.

An odd number of reflections changes the orientation of the plane to its opposite one. An even number of reflections preserves the orientation of the plane.

Thus a *movement* or *displacement* which is the mapping obtained as a result of an *even number of* successive reflections retains the orientation of the plane A skew-isometry which is the succession (or composition) of an *odd number* of reflections changes the orientation to its opposite one.

EXERCISE 2.1

1 Mark 3 points A, B, C on a circle. Place a mirror along the line l, (Fig 64). Starting from A, and looking only at the image of the circle in the mirror, trace the circular path ABC You will find a lot of fun in this. Explain why it is misleading.

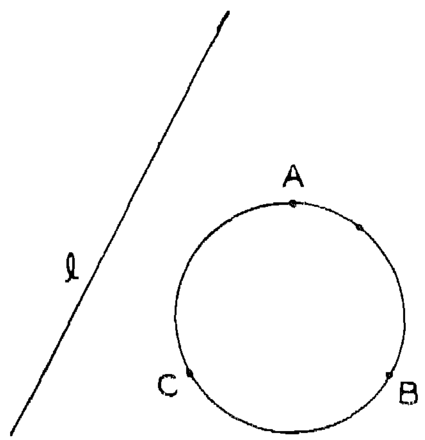


Fig. 64

2. Looking into the mirror write your name so that you can read it only through the mirror.

3. Write your name so that it will appear all right when seen in a mirror. Is the corresponding mapping a movement ?

Quadrant :—

Two mutually perpendicular lines divide the plane into 4 convex regions. Each region is called a *quadrant*.

Orientation can be explained in the following manner also. Draw two directed lines perpendicular to each other, one running from the direction West to East and the other from South to North, as shown in Figure 65.

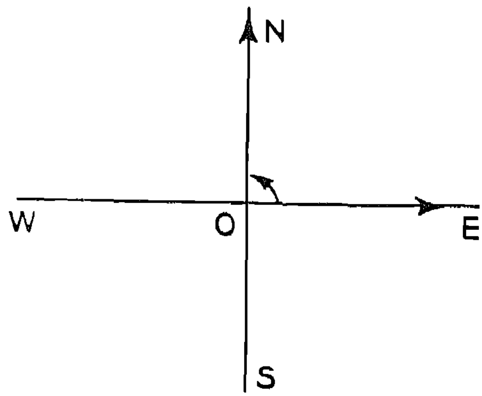


Fig. 65.

The orientation from East to North *along the quadrant* is the orientation indicated in Figure. 65. Hence orientation is determined

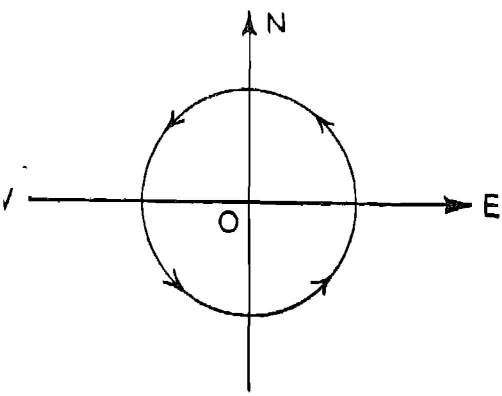


Fig. 66.

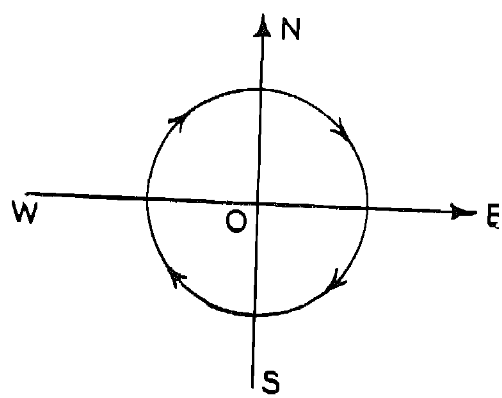


Fig. 67.

precisely (without reference to any other object like a clock et cetera) by two mutually perpendicular rays OE and ON *in order.* The pair of rays (ON, OW), (OW, OS), and (OS, OE) also give the same orientation, as shown in Figure 66.

The opposite orientation is given by any one of (ON, OE), (OE, OS), (OS, OW) and (OW, ON) as shown in Fig. 67. If we fix one of these as the orientation of the plane, then this orientation is called positive, and the other one negative.

A paradox

Stand before a mirror which is hung on the wall. Why is it that your left and right sides *are interchanged* in the reflection ; but the top and bottom *are not* ?

On the other hand, suppose you are standing on a well polished floor, or near the bank of a lake in which the water is still, and see your reflection ; or much better, see the reflection of another person. How is the image now ? It is both *upside down* and *with its right and left interchanged.* A similar thing happens if you see your image in a well polished roof. Can you explain this ?

2.3 Angle

Definitions : Draw two rays *1* and *2* from the common initial point O. The figure formed by them is called the *angle formed by the rays 1 and 2* (Fig. 68). An *angle* is the figure formed by two rays

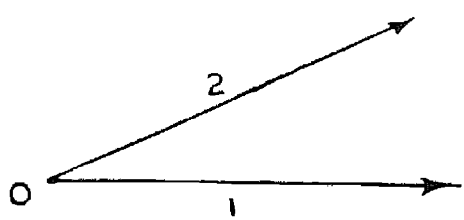


Fig. 68.

with a common initial point. The point O is called the *vertex* of the angle and the two rays *1* and *2* are called the *sides* or *arms* of the angle.

Notation: Usually an angle is denoted by one capital letter, e.g., $\angle A$ or $\angle B$ of Figure 69, where A or B is the vertex of the angle.

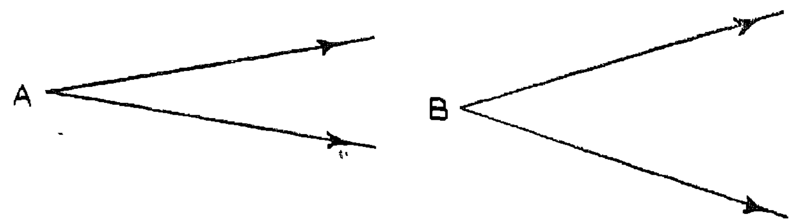


Fig. 69.

It is also denoted sometimes by a small letter or a digit written as in Figure 70.

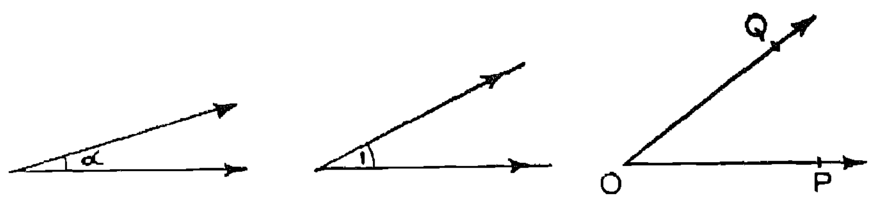


Fig. 70.

If O is the vertex and P, Q are any two points, one on each arm of the angle, then the angle is also denoted by $P\hat{O}Q$.

In Figure 71, $P\hat{O}Q$ and $R\hat{O}S$ denote the same angle.

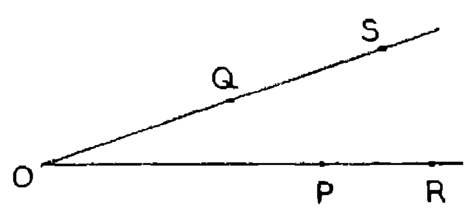


Fig. 71.

Angular Region: The full line of the ray *1* divides the plane into two half planes (Figure 72). One of these contains the ray *2*. Call

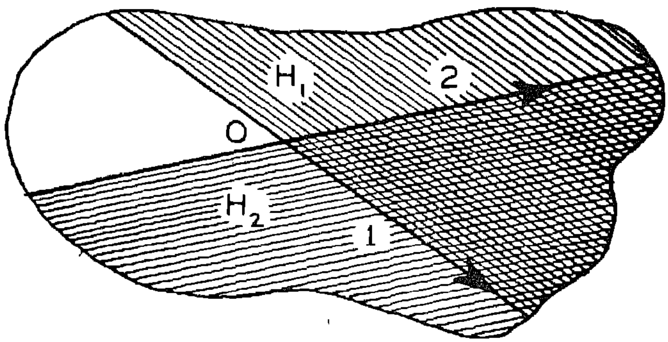


Fig. 72

this half plane H_1. Similarly the ray *2* divides the plane into two half planes. One of these contains the ray *1*. Call this H_2. Then $H_1 \cap H_2$, that is, the common part of the two regions H_1 and H_2 is called the *angular region,* formed by the rays *1* and *2* or the *interior of the angles* formed by the rays *1* and *2*.

Thus in Figure 73, the shaded part is the angular region formed by the two rays *1* and *2*. This is the interior and the unshaded part is

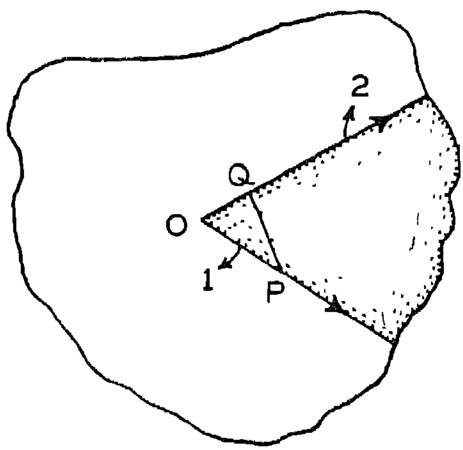


Fig 73.

the exterior of the angle. *If P is a point on the ray 1 and Q on the ray 2, then the open segment PQ lies in the angular region of 1 and 2.*

EXERCISE 2.2

1. In what other ways can you name the angle A of Figure 74?

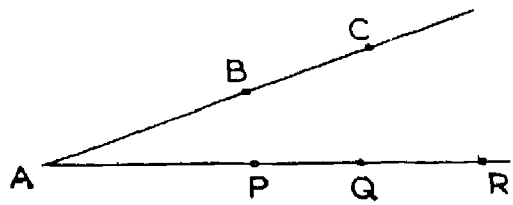


Fig. 74.

2. Name the regions in which the points A, B, C lie in Figure 75.

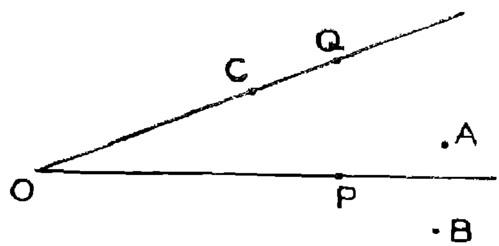


Fig. 75.

3. In Figure 76, name the regions to which the segments AB, CD, PQ and LM belong?

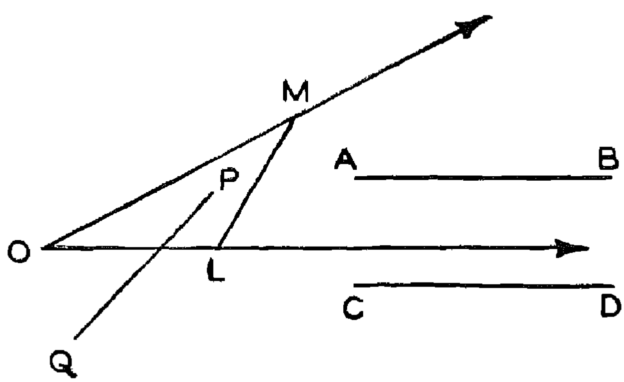


Fig. 76.

4. Name the angles formed at O in Figure 77. How are the rays *1*, *2*, *3*, *4*, *5*, situated with reference to these angles (given that *4* is parallel to *2*)?

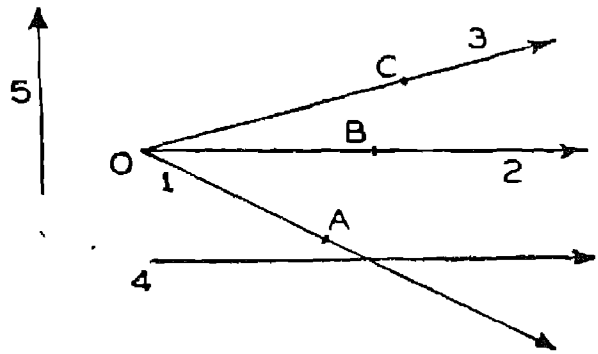


Fig. 77

5. Shade the region which is the common part of the interior of the angles $\angle ABC$ and $\angle EDC$, (Figure 78).

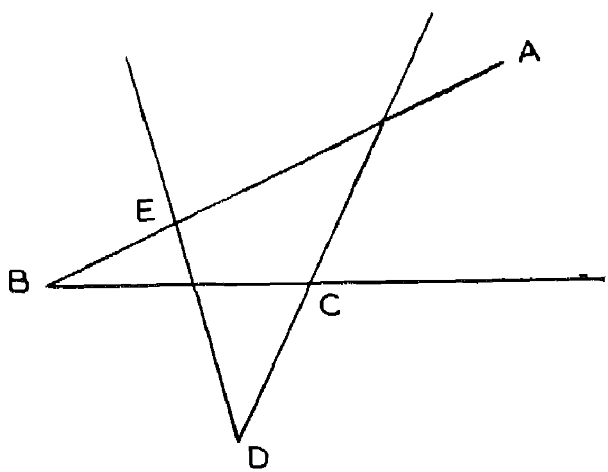


Fig. 78.

6. How many angles are there with each of the vertices A, B, C, D, E, P, Q, R, S, T ? Which are they ? Which are the angular regions to which the shaded part belongs (Fig. 79)?.

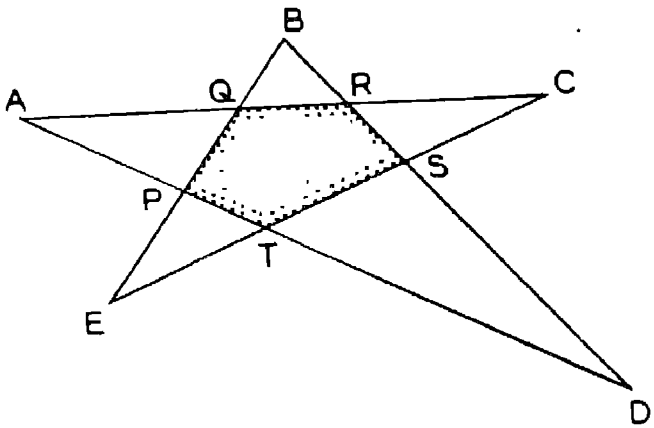


Fig. 79.

7. XOX′ and YOY′ are two perpendicular lines, dividing the plane into 4 quadrants *1*, *2*, *3* and *4*. Identify them as the angular regions of the various angles (Figure 80).

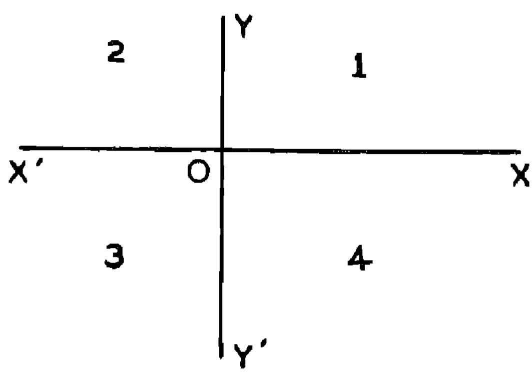


Fig. 80.

Angles of a triangle : A, B, C are 3 points that do not belong to the same line. They form a triangle. It has the three segments (also called sides) AB, BC, CA and three angles $A\hat{B}C$, $B\hat{C}A$, $C\hat{A}B$. Briefly these angles are denoted by B, C, A respectively.

Look at the point P in Figure 81.

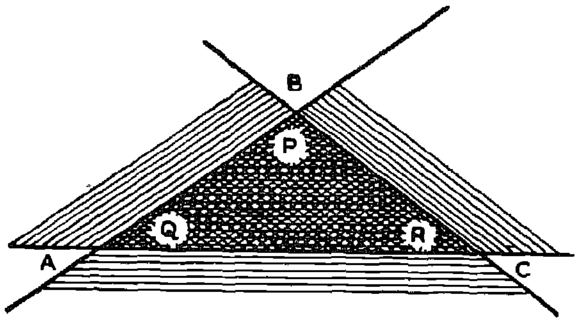


Fig 81.

It belongs to the angular regions of all the three angles A, B, C: The set of all such points P, Q, R etc., can be defined as the *inner region of the triangle*. It is therefore the common part of the angular regions of A, B and C. The open segment BC (that is, except for the end points B and C) belongs to the angular region of A.

Therefore we say that

BC is the side opposite to A
or A is the vertex opposite to BC,
CA is the side opposite to B
or B is the vertex opposite to CA
and AB is the side opposite to C
or C is the vertex opposite to AB.

2.4 Oriented Angle

Take a point P on the ray *1*, and Q on the ray *2*. The orientation of the plane which corresponds to the order O P Q is indicated along the

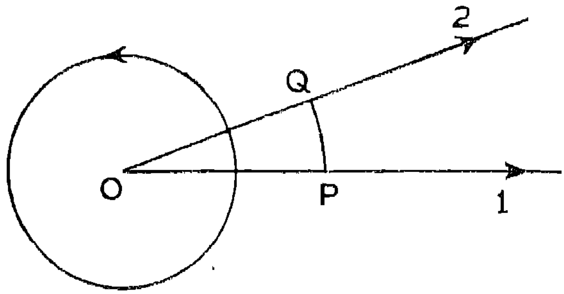


Fig. 82.

circle round O in Figure 82. This is called the orientation of the angle from the direction *1* to the direction *2*.

If this orientation is the same as the orientation which is initially fixed as positive in the plane, then the angle from *1* to *2* is called a *positive angle* or *positively* oriented angle and is written as $\angle$POQ or *the angle (1, 2)*. In this case the angle from *2* to *1* is called the *negative angle* or the *negatively oriented* angle. This is written as $\angle$QOP or the angle *(2,1)*.

We indicate the orientations of angles as shown in Figure 83.

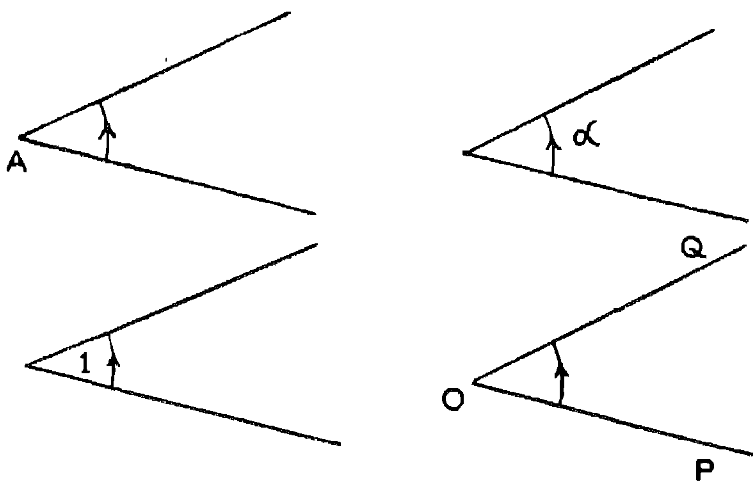


Fig. 83.

When no particular orientation is chosen in the plane, we simply call it the angle between the rays OP and OQ and we write it as either $P\hat{O}Q$ or $Q\hat{O}P$.

Example :—

In the triangle ABC, $\angle ABC$, $\angle BCA$, $\angle CAB$ are oriented in the same sense, viz., anti-clockwise. The three angles $\angle CBA$ $\angle BAC$, and $\angle ACB$ are oppositely oriented, viz. clockwise, (Figure 84).

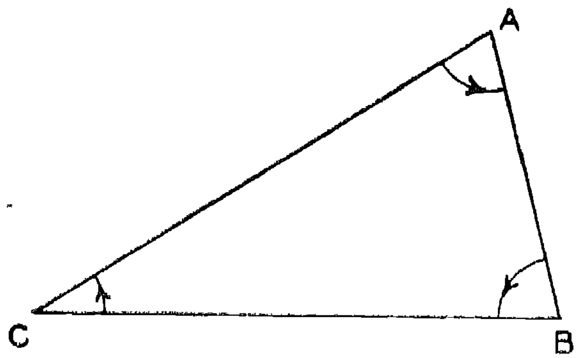


Fig 84.

EXERCISE 2.3

1. Name all the angles in Figure 85 in any one of the ways. List separately those that are clockwise oriented and those that are counterclockwise oriented.

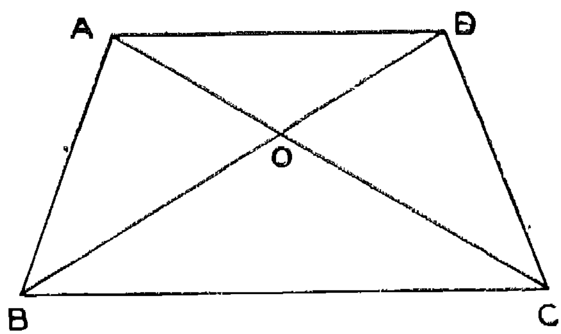


Fig. 85

2. The orientation of the plane is fixed to be (OE, ON) as shown in Figure 86. State which of the following angles are positively oriented and which negatively oriented:

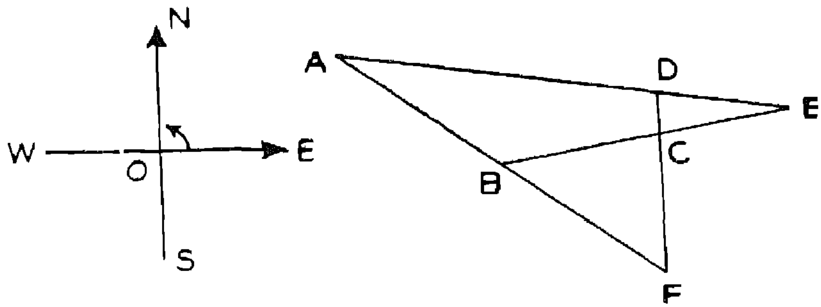


Fig. 86.

$\angle$ABC, $\angle$ADC, $\angle$AEC, $\angle$ECF, $\angle$EBF, $\angle$EDF, $\angle$EAB and $\angle$EBA.

3. A ball is passed from the player X to the player Y through the players A, B, C, D, E along the path indicated in Figure 87. Compare the orientation of the angles of deviation (that is angles

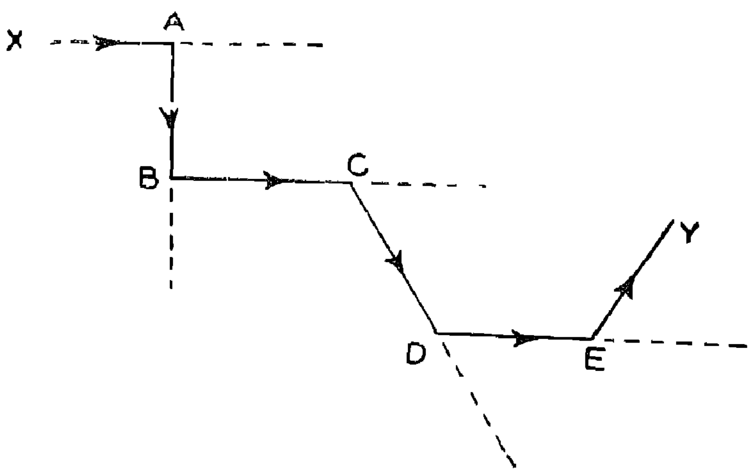


Fig. 87.

from the original direction shown in figure in dotted lines to the new direction) of the ball at the points A, B, C, D, E.

Reflection of an oriented angle :—

Study the following figures carefully, (Figure 88).

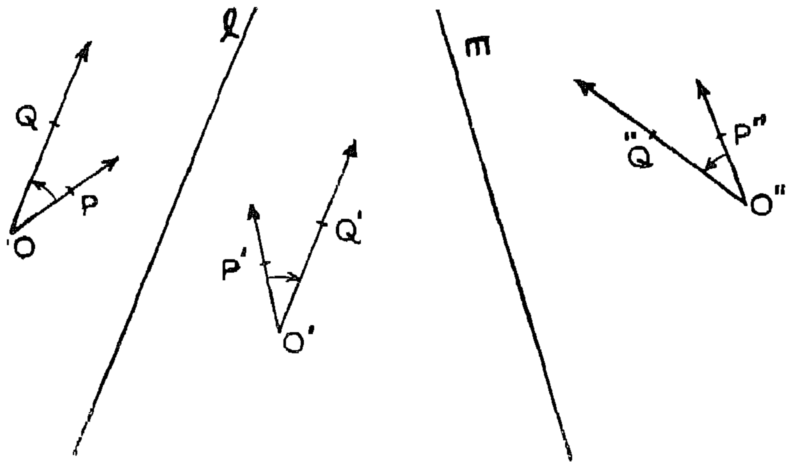


Fig. 88.

$\angle$POQ is an anticlockwise oriented angle. *l* is any line in the plane. Reflect the angle in *l*. P is mapped onto P′, O to O′ and Q to Q′ by reflection in *l*, that is, $\angle$POQ to $\angle$P′O′Q′. But $\angle$P′O′Q′ is clockwise oriented as shown by the arrow in the figure. That is, $\angle$POQ and $\angle$P′O′Q′ which are reflections of each other in the line *l* are oppositely oriented.

This is evident because, as we have already seen in the last article, *reflection in a line changes the orientation.*

Next reflect $\angle P'O'Q'$ again in another line m. Let $\angle P''O''Q''$ be the reflection of $\angle P'O'Q'$ in m This second reflection changes the orientation back to the original. Therefore $\angle POQ$ and $\angle P''O''Q''$ are similarly oriented. Also $\angle P'O'Q'$ is oppositely oriented to $\angle POQ$.

An odd number of reflections changes the orientation of an angle to its opposite one. An even number of reflections preserves the orientation of an angle.

Example :—

We have an angle (l, m) (Fig. 89). Reflect it in the line of the ray l. Reflection of l is l itself. Let m be mapped on to the ray n

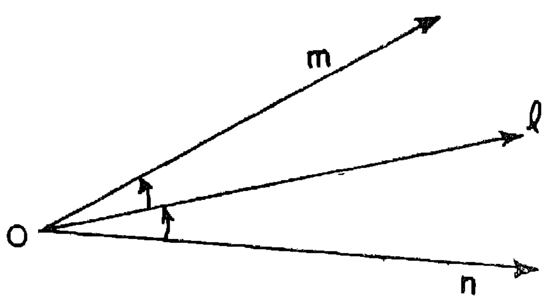


Fig. 89.

(with the same initial point O). Therefore angle (l, m) is mapped on to angle (l, n) by the single reflection. Therefore they are oppositely oriented. That means angle (l, m) and angle $(n, l,)$ are similarly oriented. m and n are mapped on to each other by the reflection in the line of the ray l. This line l is the symmetry line of the two rays m and n.

2.5 Translation Mapping

You have learnt about the fundamental mapping that is, the symmetry mapping or the reflection mapping of the plane on itself with respect to a line. Another important mapping is the *Translation* or the *Translation - Mapping* which you are going to learn now.

> A mapping of the plane in which points of the plane move in the same direction and by segments all congruent to the same segment is called a translation.

Thus a translation is completely specified if we know the direction and the segment of the translation, which is called the *vector of the translation*.

If S is a given figure, then every point of S gets displaced in the direction of translation, (Figure 90).

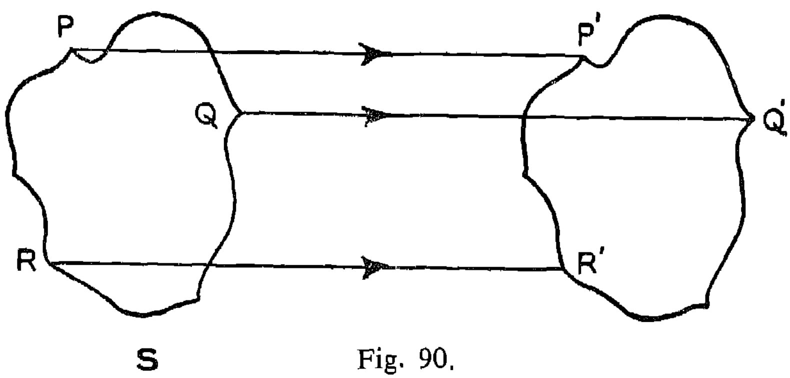


Fig. 90.

If P', Q', R' .... are the new positions of the points P, Q, R.. ... of the figure S, after translation, $PP' \simeq QQ' \simeq RR' \simeq \cdots\cdots$ etc. Also $PQ \simeq P'Q'$, $QR \simeq Q'R'$, etc., and PP', QQ', RR' ....are mutually parallel.

2.6 Translation as The Resultant of Two Successive Reflections in Two Parallel Lines

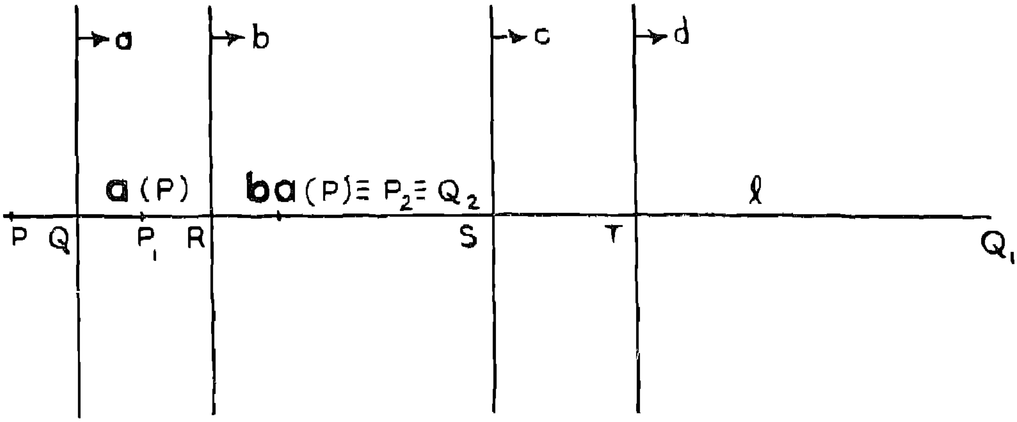


Fig. 91.

Let a and b be the two parallel lines, (Fig. 91) and P any point in the Plane. If $P_1 = \boldsymbol{a}(P) =$ the reflection of P in a then the line $[P, \boldsymbol{a}(P)]$ will be $\perp$ to a and b also. Let the line cut a in Q and b in R. Clearly $PQ \simeq Q\,\boldsymbol{a}(P)$. If $\boldsymbol{ba}(P) = P_2$ is the reflection of $\boldsymbol{a}(P)$ in the line b then $\boldsymbol{a}(P)\,R \simeq R\,\boldsymbol{ba}(P)$.

$$\text{That is,} \quad \left.\begin{array}{l} \overrightarrow{PQ} \simeq \overrightarrow{QP_1} \\ \overrightarrow{P_1R} \simeq \overrightarrow{RP_2} \end{array}\right\} \Rightarrow \overrightarrow{PP_2} \simeq 2.\overrightarrow{QR}.$$

This can be verified by means of dividers. Suppose c and d are two other lines such that each is parallel to a and b. Let the line QR intersect c and d at S and T and $\overrightarrow{QR} \simeq \overrightarrow{ST}$. Then the reflection of P first in c and then in d carries P to the same point P_2. Thus the translation $\overrightarrow{PP_2}$ determined by a and b is the same translation which is determined by c and d. The translation depends upon the direction of $\overrightarrow{PP_2}$ and the perpendicular segment $\overrightarrow{QR}$ (or $\overrightarrow{ST}$) between the parallel lines a and b (or c and d). The segment by which each point is translated is congruent to twice the perpendicular segment between the parallel lines.

Example :— Figure 92 illustrates a translation mapping.

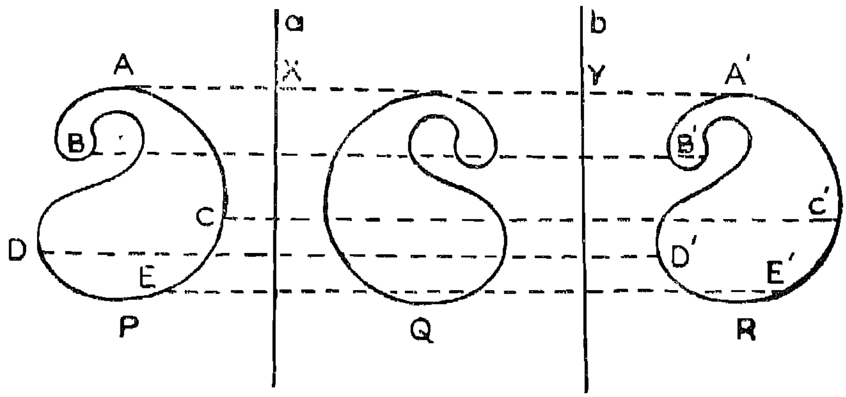


Fig. 92.

The reflection in a maps the figure P on to the figure Q and the reflection in b maps the figure Q on to the figure R. The translation $\boldsymbol{ba}$ maps P to R. AA′, BB′, CC′,....are all mutually parallel and each of them is perpendicular to the lines a and b,

$$|AA'| = |BB'| = \ldots. = 2|PQ|.$$

Verification by Geometrical Instruments

Draw any two parallel lines, a and b (Figure 91) Draw a line $\parallel$ to these and cutting a and b at Q and R. Mark a segment $\overrightarrow{ST} \simeq \overrightarrow{QR}$ on the line l, with compasses. Draw c and d each parallel to a through the points S and T. Take any point P on l. Step off a segment $\overrightarrow{QP_1} \simeq \overrightarrow{PQ}$, step off another segment $\overrightarrow{RP_2} \simeq \overrightarrow{P_1R}$ on the line l. Step off a segment $\overrightarrow{SQ_1} \simeq \overrightarrow{PS}$; step off another segment $\overrightarrow{TQ_2} \simeq \overrightarrow{Q_1T}$ You will find that Q_2 is the same point as P_2. This verifies that reflection in a and then in b gives the same translation as reflection in c and then in d.

We know that a reflection carries two parallel lines to two parallel lines, and two perpendicular lines to two perpendicular lines. We have seen that a translation may be looked upon as a 'succession' of two reflections in two parallel lines, each perpendicular to the direction of the translation. Hence properties such as, collinearity, concurrence, perpendicularity and parallelism are preserved in a translation also.

If three points A, B, C lying on a line are mapped on to the 3 points A′, B′, C′ by the translation, then the three points A′, B′, C′ also lie on a line [Figure 93 (a)].

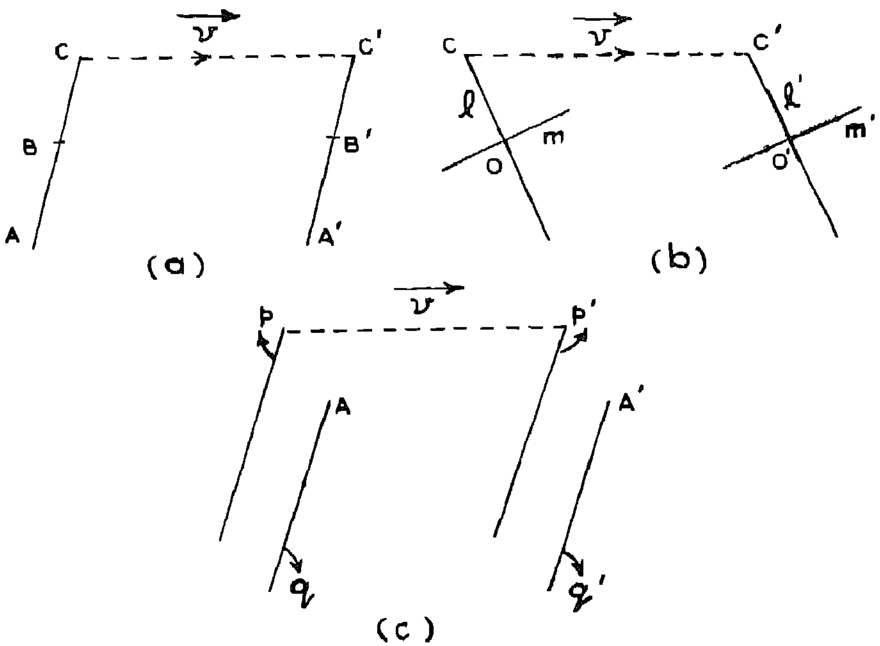


Fig 93 (a), (b) and (c)

Let the two perpendicular lines l and m be translated to the two lines l' and m'. Then l' and m' also will be perpendicular to each other [Figure 93 (b)].

Two parallel lines will remain parallel even after a translation. For example in Figure 93 (c), p//q. On translating along $\vec{v}$, if p′, q′ are their images, we have p′//q′.

(Note : Translation should be demonstrated by actual movement and by using ornamental border and other designs).

All points of a line *l* are fixed points for the reflection mapping in the line *l* ; *l* and any line perpendicular to *l* are the fixed lines. But in a translation mapping all points move in the same direction by congruent segments. No point will be mapped on to itself. Therefore *there is no fixed point in the translation mapping.*

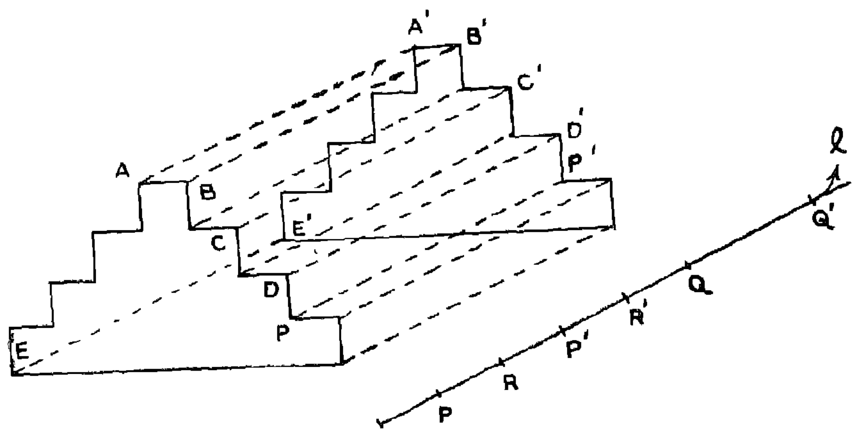


Fig. 94.

But let *l* be any line parallel to the direction of translation and P, Q, R,..... points on *l*. Then P, Q, R,....are moved along the line *l* itself to the points P′, Q′, R′,....(Figure 94), so that

$$\overrightarrow{PP'} \simeq \overrightarrow{QQ'} \simeq \overrightarrow{RR'}$$

that is, the line *l* is mapped on to itself.

Hence every line parallel to the direction of the translation is a *fixed line* of the translation mapping. It is easily seen that any other line is mapped on to a parallel line. This will be shown in class 7.

To get the translation mapping of a figure draw lines through the various points A, B, C, D etc., on the figure parallel to the direction of translation with a parallel ruler. On these lines step off congruent directed segments $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, '....each congruent to the directed segment of the translation Then A′, B′, C′...are the corresponding points on the image, (Figure 94).

2.7 Division of a Segment into Any Number of Congruent Parts

You have learnt in the previous year how to divide a given segment into any number of congruent parts. If AB is the given segment, and it is required to divide it into 6 equal parts, we take a ray AX, through A, as shown in Figure 95.

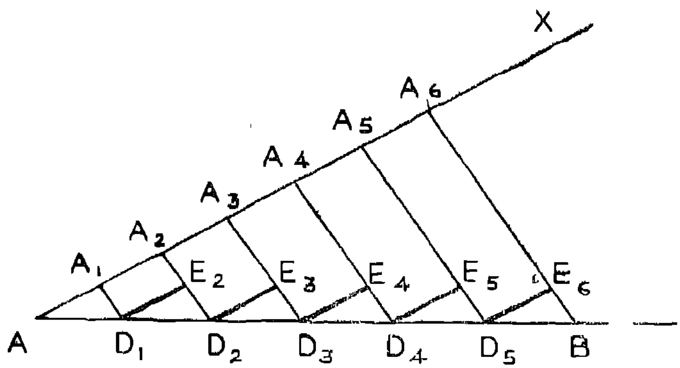


Fig 95.

Now we mark on AX, 6 congruent segments $AA_1 \cong A_1A_2 \cong \ldots\ldots \cong A_5A_6$ using a pair of compasses. We join A_6 to B, and through $A_5, A_4, \ldots\ldots, A_1$ draw parallels to A_6B, so as to meet AB in the points $D_5, D_4, \ldots\ldots, D_1$.

Let D_1E_2, $D_2E_3, \ldots\ldots, D_5E_6$ be all parallel to AX, meeting A_2D_2 in E_2, A_3D_3 in $E_3, \ldots\ldots, A_6B_6$ in E_6 respectively, as shown in Fig. 95. If we look at the triangles AA_1D_1, $D_1E_2D_2$, $D_2E_3D_3, \ldots\ldots$

we find that $D_1E_2D_2$ can be obtained from AA_1D_1 by translating it along AB, through a segment congruent to $\overrightarrow{AD_1}$. Actually AD_1 goes over to D_1D_2. Therefore D_1D_2 is congruent to AD_1

Similarly, we can get the triangle $D_2\,E_3\,D_3$ by translation of the triangle AA_1D_1 along AB through a segment congruent to D_1D_3.

AD_1 goes over to D_2D_3. Therefore D_2D_3 is congruent to AD_1.

Thus it appears that each of the segments D_1D_2, D_2D_3......, D_5B is congruent to AD_1. Hence the construction.

2.8 Congruence

The characteristic feature of elementary Geometry is the concept of *Congruence* and *Isometry* which are defined by means of reflections and explained below :

> Two plane geometrical figures are said to be *congruent* to each other if one can be mapped on to the other by successive reflections in an *even* number of lines of the plane. Every figure is congruent to itself.

We take up in detail the study of congruence of simple geometric figures.

Congruence of Angles

Two oriented angles are *congruent* to each other if one can be mapped on to the other by an even number of reflections. And we denote it by writing $\angle ABC \simeq \angle DEF$, for two congruent angles $\angle ABC$ and $\angle DEF$.

Congruence of triangles

$\triangle$ is the symbol for a triangle. When the two triangles $\triangle ABC$ and $\triangle DEF$ are congruent, we write

$$\triangle ABC \simeq \triangle DEF, \text{ in symbols.}$$

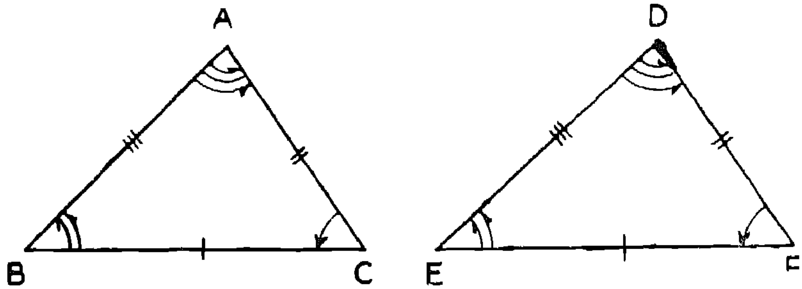


Fig. 96.

It implies the following 6 statements, (Fig. 96).

$AB \simeq DE$, $BC \simeq EF$, $CA \simeq FD$,

$\angle ABC \simeq DEF$, $\angle BCA \simeq \angle EFD$, $\angle CAB \simeq \angle FDE$.

The symbols adopted in the figure describe the situation fully.

29 Bisector of an Angle

Let *1* and *2* be two rays with the common initial point O. We know that there is exactly one line of symmetry *l* of the two rays, *1* and *2*, (Figure 97 (a)).

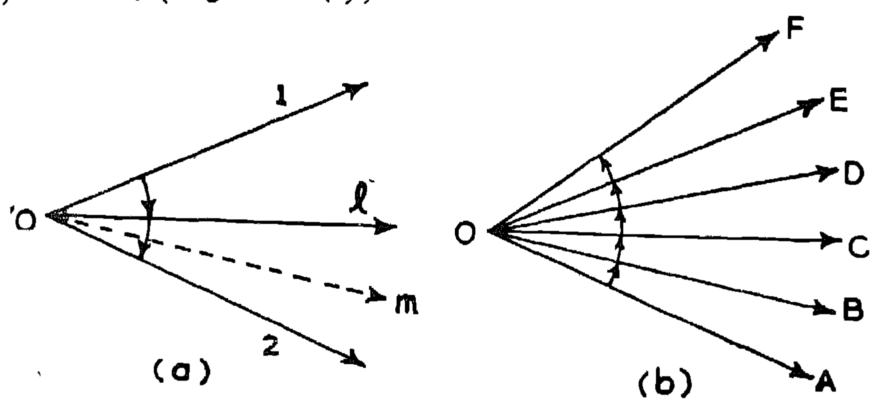


Fig. 97 (a) and (b)

By reflection in l, 1 is mapped on to 2 and l to itself. Therefore angle $(1, l)$ is mapped onto angle $(2, l)$. This is written in symbols as $\boldsymbol{l}$ [angle $(1, l)$] = angle $(2, l)$.

Let $\boldsymbol{m}$ be the symmetry line of 2 and l.
That is reflection in m maps 2 to l and l to 2.
That is $\boldsymbol{m}$ maps angle $(2, l)$ onto angle $(l, 2)$, or
$\boldsymbol{m}$ [angle $(2, l)$] = angle $(l, 2)$.
That is $\boldsymbol{l}$ maps angle $(1, l)$ on to angle $(2, l)$ and $\boldsymbol{m}$ maps angle $(2, l)$ onto angle $(l, 2)$.
The two reflections first $\boldsymbol{l}$ and then $\boldsymbol{m}$ map the angle $(1, l)$ on to the angle $(l, 2)$.
Therefore they are congruent.

$$\text{Angle } (1, l) \simeq \text{Angle } (l, 2).$$

They are represented by the same type of arrows in the figure. Hence l, the symmetry line of the rays 1 and 2 is called the *Bisector* or the *symmetry line of angle* $(1, 2)$.

Example :— Let OA, OB be any two rays from O, Figure 97 (b). Let OC be the reflection of OA in the line of OB. That is OB is the line of symmetry of OA and OC. Therefore

$$\angle AOB \simeq \angle BOC \qquad \text{......(1)}$$

Next let OD be the reflection of OB in the line of OC, that is $\angle BOC \simeq \angle COD$ and so on. Therefore this gives a method of constructing $\angle BOC$, $\angle COD$, $\angle DOE$ etc each congruent to the given $\angle AOB$.

2.10 An Important Construction

Suppose we are given (an oriented angle) $\angle POQ$ and a ray O′X. We will now give a method of constructing $\angle XO'Q'$ congruent to $\angle POQ$ by successive reflections in two lines that is $\angle POQ$ and $\angle XO'Q'$ will be congruent with the same orientation. Let a be the

line of symmetry of O and O′ (Figure 98). Reflect in a. That is fold the paper along a.

By means of this mapping denoted by $\boldsymbol{a}$, the figure POQ is mapped on to the figure $P_1O'Q_1$ as shown in the Fig. 98.

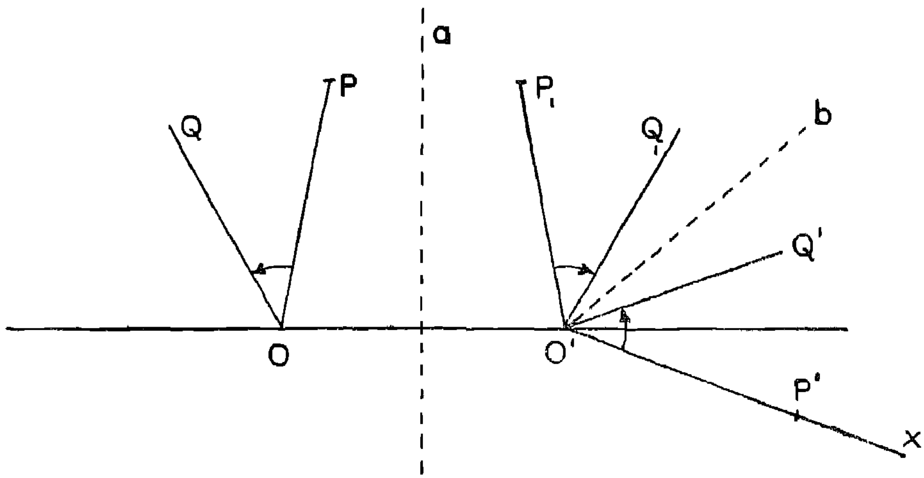


Fig 98.

As before we denote this in symbols as :

$$\boldsymbol{a}\,(POQ) = P_1O'Q_1.$$

This means the mapping $\boldsymbol{a}$, that is reflection in 'a' maps POQ to $P_1O'Q_1$. Let b be the symmetry line of the two rays $O'P_1$ and O′X. Now reflect again in b. Call this mapping $\boldsymbol{b}$. That is fold the paper again along b. This maps $O'P_1$ to O′P′ which will be along O′X, let $O'Q_1$ be mapped on to O′Q′. *Thus the mapping* $\boldsymbol{b}\,\boldsymbol{a}$ *which means reflection first in* 'a' *and then in* 'b' *maps* $\angle POQ$ *on to* $\angle P'O'Q'$ *which is the same as* $\angle XO'Q'$.

Therefore $\angle XO'Q' \simeq \angle POQ$.

We have $\boldsymbol{ba}\,(POQ) = P'O'Q'$.

That is $\boldsymbol{ba}$ maps the figure POQ onto the figure P′O′Q′.

And $\triangle\ POQ \simeq \triangle\ P'O'Q'$.

2.11 Some Definitions

1. Let OA, OB, OC be three rays from O. The two angles $\angle AOB$ and $\angle BOC$ have a common arm OB, such that OA, OC are *on opposite sides of OB*. Two such angles are called *adjacent angles*, (Figure 99).

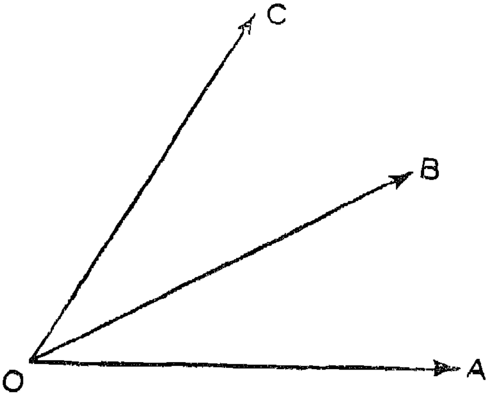


Fig. 99.

OC belongs to the exterior region of $\angle AOB$ and OA belongs to the exterior region of $\angle BOC$; OB belongs to the interior of $\angle AOC$ (except for the point O).

2. A′A, and B′B are two parallel lines, (Figure 100). PQ is any transversal that is a line intersecting A′A and B′B at X and Y.

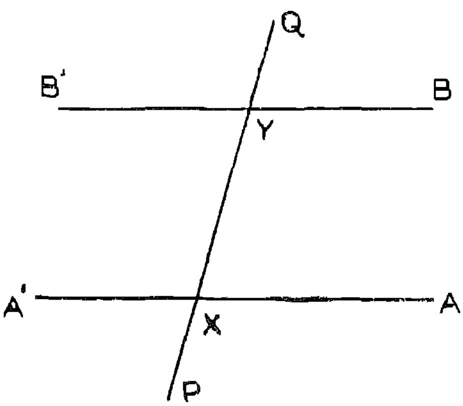


Fig. 100.

$\angle AXQ$ and $\angle BYQ$ are called a pair of *corresponding angles*. ($\angle AXP$, $\angle BYP$), ($\angle A'XQ$, $\angle B'YQ$) and ($\angle A'XP$, $\angle B'YP$) are also pairs of corresponding angles.

($\angle AXQ$, $\angle B'YP$) and ($\angle A'XQ$, $\angle BYP$) are two pairs of *alternate angles*.

3. AOA′, BOB′ are two intersecting lines, (Figure 101). Then $\angle AOB$ and $\angle A'OB'$ are *vertically opposite angles*.

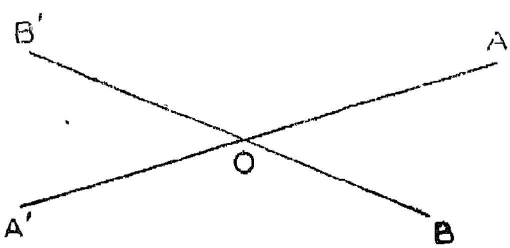


Fig. 101

$\angle AOB'$ and $\angle A'OB$ form another pair of vertically opposite angles.

This can be demonstrated by paper folding. Fold a sheet of paper along two lines POP′, QOQ′. Fold again such that the ray OP falls along OQ′. By this folding OQ falls along OP′ This verifies that $\angle$POQ and $\angle$Q′OP′ are obtainable from each other by a single folding that is, a single reflection, that is, they are oppositely oriented.

Therefore. $\angle POQ \simeq \angle P'OQ'$.

If we fold such that OP falls along OQ, then OQ′ falls along OP′. That is, $\angle$POQ′ and $\angle$QOP′ are oppositely oriented.

Therefore: $\angle POQ' \simeq \angle P'OQ$.

2 14 Symmetry-line of Two Parallel Lines — Rectangle

You have learnt that a pair of intersecting lines AOB and COD have two lines of symmetry namely the pair of bisectors of the angles between the lines.

Suppose we have two parallel lines a and b; let p be any line which is perpendicular to the line a. We show that p is perpendicular to the line b also (Figure 104).

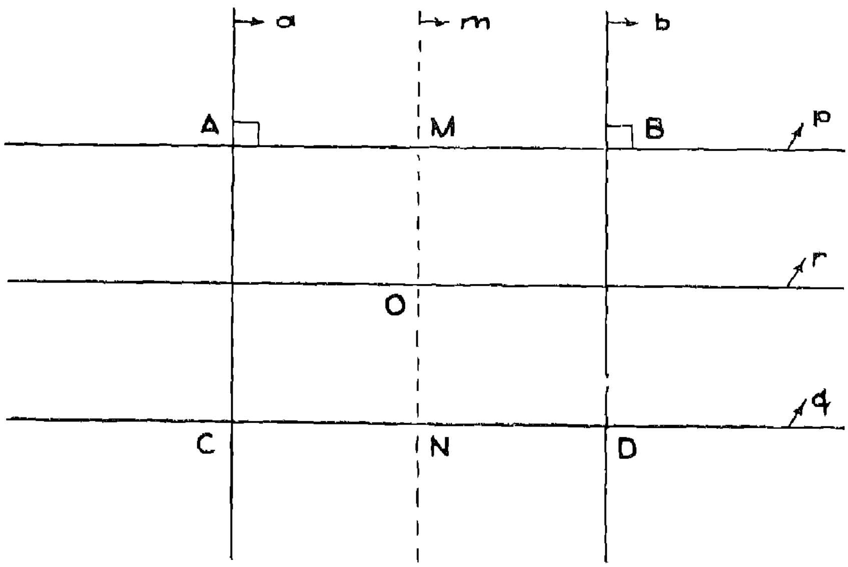


Fig 104

2.13(a).

> Alternate angles made on two parallel lines by a transversal are congruent.

Let the two lines AA′, BB′ be cut by the transversal PQ at X and Y, Fig. 103 (a).

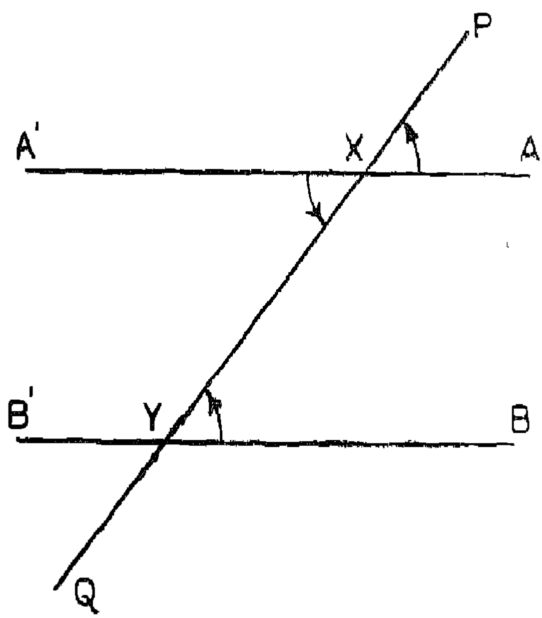


Fig. 103(a).

$\angle BYP \cong \angle AXP$ (corresponding angles).

$\angle AXP \cong \angle A'XQ$ (vertically opposite angles).

$\therefore \angle BYP \cong \angle A'XQ$ and they are a pair of alternate angles.

$$\left.\begin{array}{r} p \perp a \Rightarrow \text{ Angle } (a, p) = \dfrac{\pi}{2} \\ a \parallel b \end{array}\right\} \Rightarrow \text{Angle } (b, p) = \frac{\pi}{2}$$

$$\Rightarrow \quad b \perp p$$

(By congruence of corresponding angles.)

If $\boldsymbol{p}$ denotes reflection in the line p, then we have

$$\boldsymbol{p}\,(a) = a, \quad \boldsymbol{p}\,(b) = b$$

or

Any line p perpendicular to a line a is a symmetry line of each of the lines parallel to the line a

This set of lines $p, q, r, \ldots$ which are perpendicular to the lines a and b are evidently parallel to one another.

Apart from this set of lines, two lines a and b have another line of symmetry which is obtained in the following way. Let any line p which is perpendicular to a and b meet them at A and B.

i e., $p \perp a, \quad p \cap a = \text{A}, \quad p \cap b = \text{B}.$

Let m be the symmetry line of A and B.

Let $\qquad \text{M} = m \cap p.$

Then M is the middle point of AB.

a is the line perpendicular to p at A.

Hence $\boldsymbol{m}\,(a)$ is the line perpendicular to $\boldsymbol{m}\,(p)$ at $\boldsymbol{m}\,(\text{A})$,

i.e., $\boldsymbol{m}\,(a)$ is the line perpendicular to p at B, i.e., the line b,

$$\left[\begin{array}{l} \because \boldsymbol{m}\,(p) = p, \qquad \boldsymbol{m}\,(\text{A}) = \text{A} \\ \text{and reflection preserves perpendicularity} \end{array}\right]$$

or $\quad\quad \boldsymbol{m}(a) = b$ and $\boldsymbol{m}(b) = a$

that is, m is also a symmetry line of the two parallel lines a and b.

And

$$\left.\begin{matrix} m \perp p \\ p \perp a,\ p \perp b \end{matrix}\right\} \Longrightarrow m \parallel a,\ m \parallel b$$

Next suppose another line q be perpendicular to a and b.

Let $\quad q \cap a = \text{C}, \quad q \cap b = \text{D}.$

$$\begin{matrix} m \parallel a \\ a \perp q \end{matrix} \Longrightarrow m \perp q \text{ or } m(q) = q$$

and

$$\text{C} = a \cap q \Longrightarrow \left\{ \begin{aligned} \boldsymbol{m}(C) &= \boldsymbol{m}(a) \cap \boldsymbol{m}(p) \\ &= b \cap q \\ &= \text{D} \end{aligned} \right\}$$

that is, m is also the symmetry line of CD. Hence we obtain

> A pair of parallel lines a and b have a line of symmetry which is parallel to each of the lines. We call this as the line of symmetry of the pair of parallel lines a and b.

EXERCISE 2.4

1. Consider the quadrilateral ABDC (Figure 104). Are all its angles right angles ? Why ?

2. Show that m is a line of symmetry of the quadrilateral ABDC. We call this a *Rectangle*. Has this figure another line of symmetry ? If so specify it.

3. What are the points $\boldsymbol{m}$ (A), $\boldsymbol{m}$ (D) ? Deduce that AD $\simeq$ BC.

4. Show that AD and BC intersect at O, the point of intersection of the two symmetry lines m and r of the quadrilateral ABDC (i.e., rectangle ABDC).

That is, $AD \cap BC = m \cap r$.

Give reasons for
i. $AO \simeq BO$
ii. $CO \simeq DO$
iii. $AO \simeq CO$

(Use that r is also a line of symmetry of the quadrilateral).

Hence you have learnt:

> The diagonals of a rectangle are equal in length and bisect one another.

5. Name the adjacent angles in the figure formed when the lines AOA′, and BOB′ intersect at O.

6. Name the adjacent angles and vertically opposite angles in Fig. 105. Which angles are congruent?

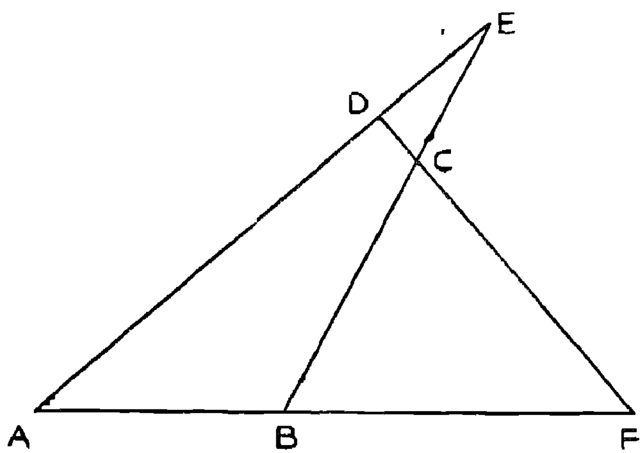


Fig. 105

7. Name corresponding angles and alternate angles in Fig. 106, (O′B′ is parallel to OA). Write all angles congruent to ∠BOA.

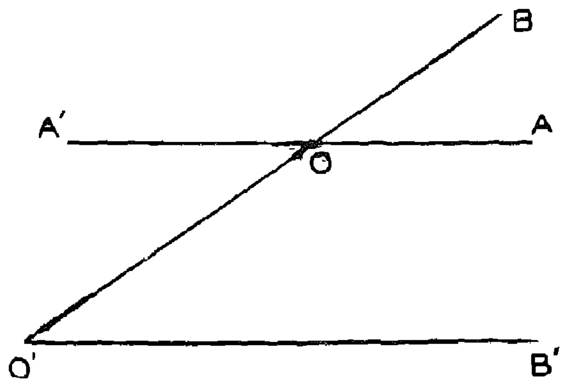


Fig. 106.

8. In figure 107 AA′ is parallel to BB′, CC′ is parallel to DD′.

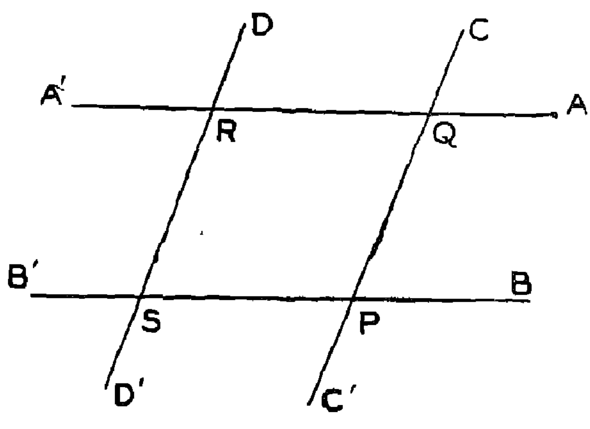


Fig. 107.

Name the following:

(a) adjacent angles of ∠AQC, ∠BPC′, ∠A′RD, ∠PSD′

(b) corresponding angles of ∠CQA, ∠BSD, ∠A′RD, ∠B′PC.

(c) vertically opposite angles of ∠CQA, ∠BSD′, ∠A′RD′ ∠BSD.

(d) Name alternate angles of $\angle$BPC, $\angle$DSB, $\angle$B′PC, $\angle$ARD′ List all angles congruent to $\angle$CQA and to $\angle$D′SB in the figure.

9. Given a sheet of paper with two rays whose common end point is outside the sheet of paper, construct by paper folding the bisector of the angle formed by the two rays.

10. What is the orientation of each angle *from* the red ray *to* the green ray at various points in the figures of the chart ?

11. Point out pairs of (a) adjacent angles, (b) vertically opposite angles (c) alternate angles and (d) corresponding angles in the figures of the chart.

12. In each figure of the chart, measure the angles and group them into sets of congruent angles. Wherever possible, give reasons for the congruence of these angles.

3

Measure of an Angle

3.1 Addition of Any Two Segments

Given any two segments AB, CD we show that it is possible to map the segment CD on to a segment BE on the ray AX. i.e , the ray of the segment $\overrightarrow{AB}$, by two successive reflections, (Fig. 108).

Let l be the symmetry line of B, C. Reflect first in l. let $l(CD) = BD'$, i.e., the segment CD is mapped on to the segment BD' (as shown in Fig. 108) by reflection in l.

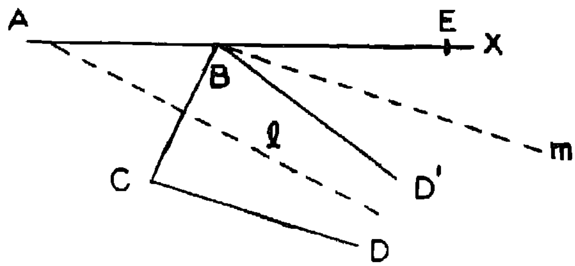


Fig 108.

Let m be the symmetry line of $\angle XBD'$, Reflect in m. BD' will be mapped on to BE, along $\overrightarrow{BX}$. Therefore reflections first in l and then in m carry CD to BE. Now we define the sum of the two segments AB and CD as the sum of the segments AB and BE i.e., AE. Therefore we can write

$$AB + CD \simeq AB + BE \simeq AE.$$

Note :— This is one of the ways of constructing $BE \simeq CD$. This may be done e.g., by reflecting in the symmetry line of BD., so that D falls at B and then reflect in a suitable line through B so

that CD falls along $\overrightarrow{BX}$ after these two reflections. That we get the same point E by this way or in some other way with an even number of reflections is the *Fundamental Axiom of Elementary Geometry.*

Addition of segments is commutative

By reflecting A,B,E in the symmetry line of AE (Fig. 108) we can show that CD + AB is also equal to AE. i.e, two segments can be added in any order.

$$AB + CD \simeq CD + AB.$$

Addition of segments is associative

If we want to add 3 numbers, first we add any two of them and to this sum we add the remaining number. Similarly if we want to add 3 segments say *a,b,c,* first we add any two of them and to this sum we add the remaining segment.

Example :—

$$a + b + c \simeq (a + b) + c$$

or

$$a + b + c \simeq a + (b + c).$$

See figure 109.

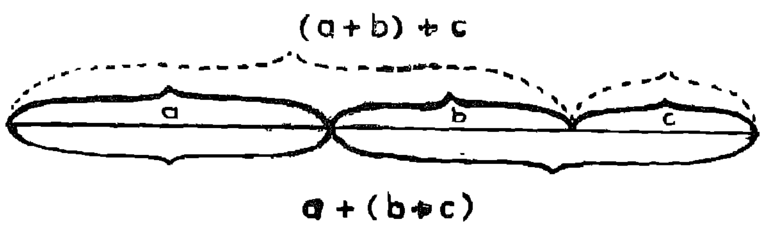


Fig. 109.

Exercise :—

Show in a figure all other possible ways of placing these segments *a,b,c* on the same line side by side for the purpose of addition.

Like this we can add any 4 or 5 or any finite number of segments, adding two at a time.

Addition of congruent segments

Suppose we have two segments a and b each congruent to OA_1. Then $a + b \simeq OA_1 + OA_1$ and we write this as 2 OA_1.

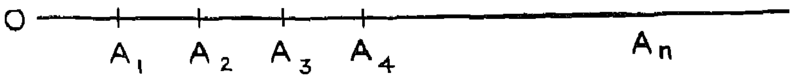


Fig. 110.

The sum of 3 segments a,b,c each congruent to OA_1 is

$$a + b + c \simeq OA_1 + OA_1 + OA_1$$

and is written as $\quad 3\ OA_1$.

Similarly the sum of n congruent segments each congruent to the segment OA_1 is

$$(OA_1 + OA_1 + OA_1 + \dots \dots + OA_1 \text{ n times}) \simeq n.\ OA_1.$$

We can represent it along the ray OA_1 like this : Mark points (Fig. 110), $A_2, A_3, A_4, \dots\dots, A_n$, on the ray OA_1 such that:

$$OA_1 \simeq A_1A_2 \simeq \dots\dots \simeq A_{n-1}A_n,$$

i.e., take n steps each congruent to OA_1 along the ray OA_1. Then the sum of these n congruent segments is $OA_n \simeq n.\ OA_1$.

3.2 Subtraction of Segments

So far, while mapping various segments along the same ray, all these were mapped on to segments oriented in the same direction. We will call the oppositely oriented segment $\overrightarrow{DC}$ as the negative of $\overrightarrow{CD}$ and write as $-\overrightarrow{CD}$. We now define subtraction as follows :—

$\overrightarrow{AB}$ minus $\overrightarrow{CD}$ is congruent to $\overrightarrow{AB}$ plus negative of $\overrightarrow{CD}$ i.e.,

$$\overrightarrow{AB} - \overrightarrow{CD} \simeq \overrightarrow{AB} + (-\overrightarrow{CD})$$

We write this as: $\quad \overrightarrow{AB} - \overrightarrow{CD}$.

To construct $a-b$, geometrically where a, b are any two segments, map the segment a on to a segment $\overrightarrow{AB}$ on a line l. Map the segment

b on to a segment $\overrightarrow{BE}$ on l, so that $\overrightarrow{AB}$ and $\overrightarrow{BE}$ are *oppositely* oriented on the line l

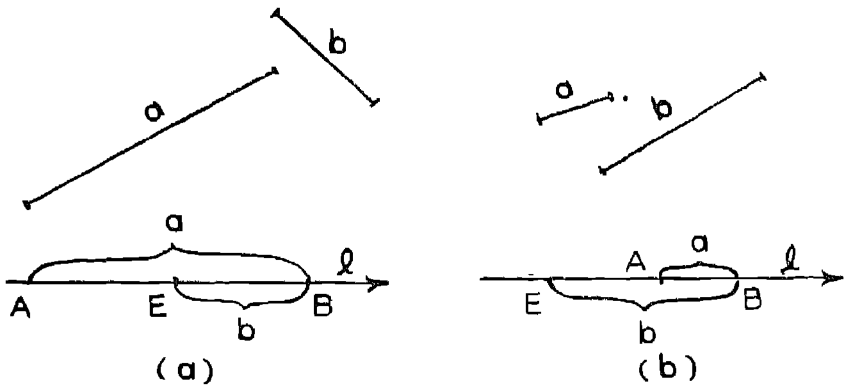


Fig 111

Now E may fall between A and B [(as in Fig. 111 (a)] or beyond A [(as in Fig. 111 (b)]. In either case we define $a-b$ as the segment AE. i.e., $a-b \simeq \overrightarrow{AB}+\overrightarrow{BE} \simeq \overrightarrow{AE}$.

If E falls on A itself, $a-b \simeq \overrightarrow{AB}+\overrightarrow{BA} \simeq \overrightarrow{AA}$.

This is called a *null segment.*

This means, if we take one step *forward* equal to a and another step *forward* equal to b, it becomes *one step forward* equal to $a+b$. If we take one step *forward* equal to a and the second step backwards equal to b, it becomes either one step *forward equal to* $a-b$ (Fig. 111 (a)) or one *step backwards equal to* $b-a$ (Fig 111 (b)).

Let A,B,C be 3 points on a line taken in any order, then always.

$$\overrightarrow{AB}+\overrightarrow{BC} \simeq \overrightarrow{AC}.$$

This is illustrated in (Fig. 112).

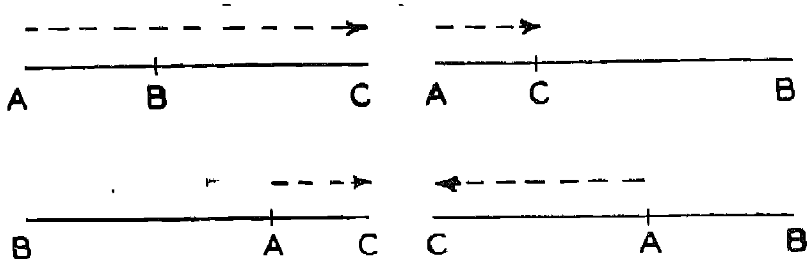


Fig. 112.

If we have a number of points, say A,B,C,D,E,F on a line placed in any order, then

$$\begin{aligned}(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} &\simeq \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} \\ &\simeq (\overrightarrow{AC} + \overrightarrow{CD}) + \overrightarrow{DE} + \overrightarrow{EF} \\ &\simeq (\overrightarrow{AD} + \overrightarrow{DE}) + \overrightarrow{EF} \\ &\simeq \overrightarrow{AE} + \overrightarrow{EF} \\ &\simeq \overrightarrow{AF}.\end{aligned}$$

Let OA_1 be a given segment on a line. Take n steps in the direction of OA_1, (Fig. 113). We get $\overrightarrow{OA_n} \simeq n.\,\overrightarrow{OA_1}$. If we take n steps $\overrightarrow{OB_1}$, $\overrightarrow{B_1B_2}$, $\overrightarrow{B_2B_3}$, ..., $\overrightarrow{B_{n-1}B_n}$ on the same line in the opposite direction, i.e., in the direction of $\overrightarrow{A_1O}$, then

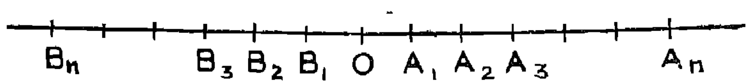


Fig. 113.

$$\overrightarrow{OB_n} \simeq -\overrightarrow{OA_n} \simeq -n.\,\overrightarrow{OA_1}$$

Here B_1, B_2, ... are reflections of A_1, A_2, ... in the point O.

EXERCISE 3.1

1. Mark points A,B,C,D,E,F in different orders on a line and verify that always $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} \simeq \overrightarrow{AF}$.

2. Verify that $m.OA + n.\ OA \simeq (m + n)\ OA$, for all positive values of m and n.

3.3 Addition of Angles

If $\angle AOB$ and $\angle BOC$ are two adjacent, oriented angles, (Fig. 114) then the sum of these two angles is defined as $\angle AOC$ and is written as:

$$\angle AOB + \angle BOC \simeq \angle AOC.$$

If $\angle AOB$ and $\angle PO'Q$ are any two angles of the same orientation, in order to add them we use two reflections (i.e., a movement or a congruent mapping) so that $\angle PO'Q$ is mapped on to $\angle BOC$ i.e.,

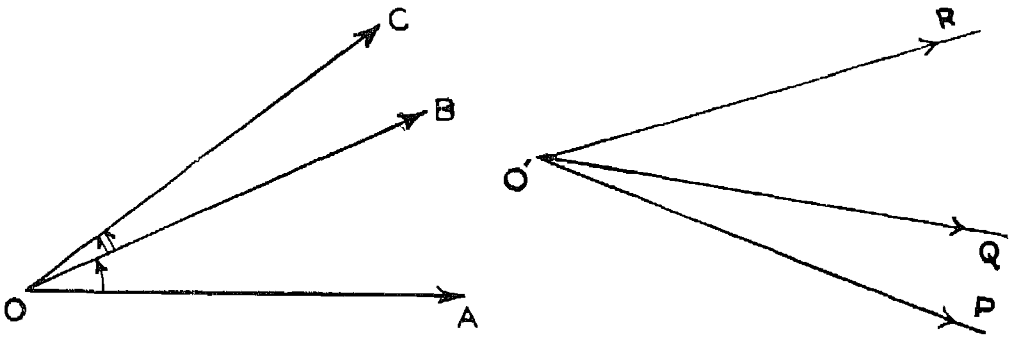


Fig. 114.

the rays O'P and O'Q are mapped on to the rays OB and OC. The rays OA and OC will be on opposite sides of the line of the ray OB. Then we define AOC as the sum of the two angles. This is written as:

$$\begin{aligned}\angle AOB + \angle PO'Q &\simeq \angle AOB + \angle BOC \\ &= \angle AOC.\end{aligned}$$

We can obtain $\angle AOC$ the sum of the two angles by rotating the ray OB through an angle congruent to $\angle PO'Q$ to the position OC. If we rotate the ray O'Q to the position O'R through an angle congruent to $\angle AOB$ then also $\angle PO'R$ gives the sum of the two angles, i.e.,

$$\begin{aligned}\angle PO'Q + \angle AOB &\simeq \angle PO'Q + \angle QO'R \\ &= \angle PO'R\end{aligned}$$

(Here the sense of the rotations must be same as the orientation of the angles). And we have $\angle AOB + \angle PO'Q \simeq \angle PO'Q + \angle AOB$ That is, the addition can be performed in any order. In other words

The addition of angles is commutative

Also

The addition of angles is Associative.

You know that we can add any number of numbers by adding only two at a time. Similarly we can add any number of angles. Let A, B, C be three angles of the same orientation. First find the sum of any two of them. To this sum add the remaining angle. This is the sum of the three angles.

$$\begin{aligned} A + B + C &\simeq (A + B) + C \\ \text{Or} &\simeq (B + C) + A \\ \text{Or} &\simeq (C + A) + B \end{aligned}$$

that is, map them side by side to form adjacent angles in any order that is, in any one of the following ways shown in Figure 115. $\angle XOY$ got in any of these ways are all congruent and give the sum A + B + C.

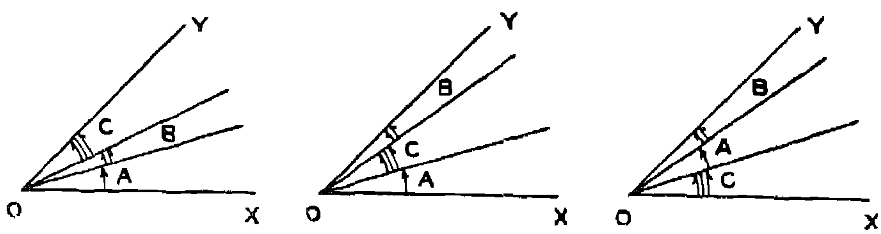


Fig. 115.

Write the other possible ways of associating the three angles and illustrate them in figures. Like this we can add any finite number of angles, adding two at a time.

Positive integral multiple of an angle

Given $\angle AOB$, we next define $2\ \angle AOB$ as $\angle AOB + \angle AOB$

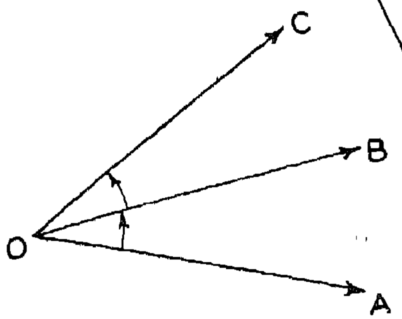


Fig. 116.

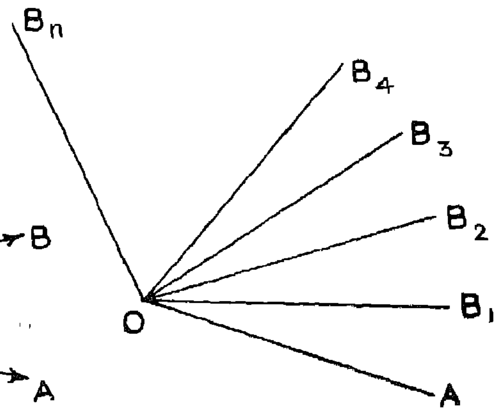


Fig. 117.

that is, in this case we make: $\angle BOC \simeq \angle AOB$ (Fig. 116). Therefore given $\angle AOB_1$, we can draw rays OB_2, $OB_3 \ldots$ such that (Figure 117).

$$\angle AOB_1 \simeq \angle B_1OB_2 \simeq \angle B_2OB_3$$

then

$$\begin{aligned} \angle AOB_2 &\simeq \angle AOB_1 + \angle B_1OB_2 \\ &\simeq \angle AOB_1 + \angle AOB_1 \\ &\simeq 2\ \angle AOB_1 \\ \angle AOB_3 &\simeq \angle AOB_1 + \angle B_1OB_2 + \angle B_2OB_3 \\ &\simeq \angle AOB_1 + \angle AOB_1 + \angle AOB_1 \\ &\simeq 3\ \angle AOB_1 \text{ and so on.} \end{aligned}$$

We have

$$\begin{aligned} \angle AOB_n &\simeq \angle AOB_1 + \angle B_1OB_2 + \ldots\ldots + \angle B_{n-1}OB_n \\ &\simeq \angle AOB_1 + \angle AOB_1 + \angle AOB_1 + \ldots\ldots + \\ \angle AOB_1 &\simeq n.\ \angle AOB_1 \end{aligned}$$

that is, if the ray OA rotates, n times through an angle congruent to $\angle AOB_1$ in the direction of the orientation of $\angle AOB_1$, it takes the position OB_n making an angle $n.\angle AOB_1$ with the ray OA.

3.4 Subtraction of Angles

Subtraction of one angle from another is defined as the addition to the second angle, of the first angle taken with the opposite orientation.

Let $\angle AOB$ and $\angle PO'Q$ be two angles. Here we define $\angle AOB - \angle PO'Q$ to be $\angle AOB + \angle QO'P$.

To construct $\angle AOB - \angle PO'Q$, by two reflections, $\angle QO'P$ is mapped on to the position $\angle BOC$ so that the ray $O'Q$ is mapped on to the ray OB and $O'P$ to OC. The ray OC may belong either to the interior (Figure 118 (a)) or to the exterior (Figure 118 (b)) of the angular region of $\angle AOB$. In either case, $\angle AOB - \angle PO'Q$ is defined as $\angle AOC$.

(If OC falls along OA, we get $\angle AOB - \angle PO'Q = \angle AOA$ which is called a *Null angle*).

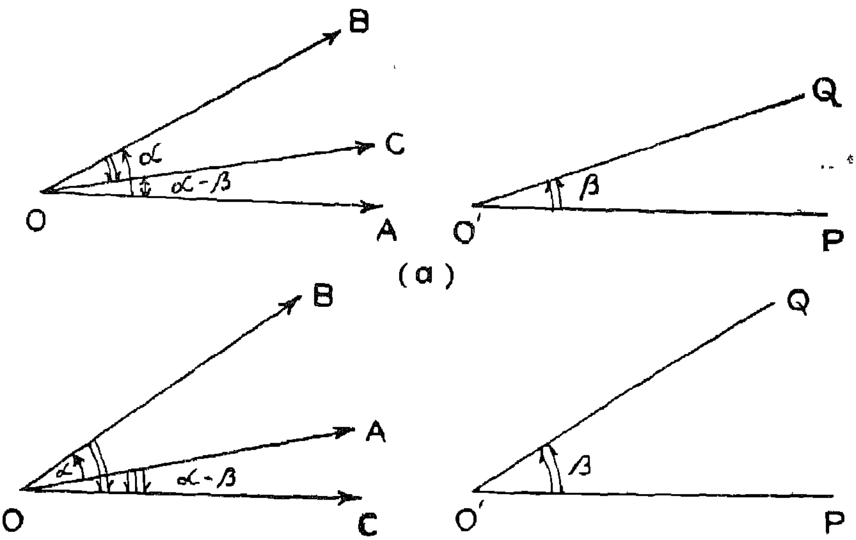


Fig. 118. (b)

In the first case $\angle AOC$ has the orientation of $\angle AOB$ and in the second case it has the opposite orientation, i. e., the orientation of $\angle QO'P$.

We can also look at it in an intuitive way like this. Given $\angle AOB$ and $\angle PO'Q$ rotate the arm OB in the direction opposite to

the orientation of $\angle AOB$ through an angle congruent to $\angle QO'P$ to take the position OC. Then we can define $\angle AOB - \angle PO'Q$ as $\angle AOC$.

We have seen here that $\angle AOB + \angle BOC$ is always congruent to $\angle AOC$. i.e., whether they are similarly oriented or oppositely oriented. *If nothing is stated about the orientation, it is understood that the given angles to be added or subtracted have the same orientation.*

Negative integral multiple of an angle

Given $\angle AOA_1$, if OA is rotated n times through an angle congruent to $\angle AOA_1$, we get the angle $\angle AOA_n$ congruent to n $\angle AOA_1$. But if OA is rotated through an angle congruent to $\angle A_1OA$, i.e., in the opposite direction, we get OB_1 such that $\angle AOB_1 \simeq \angle A_1OA$, which we call the negative of $\angle AOA_1$ and write $- \angle AOA_1$. Rotating again through the same angle we get OB_2 such that $\angle AOB_2 \simeq - \angle AOA_2 \simeq - 2 \angle AOA_1$ and so on. Therefore by rotating n times, we get, (Fig. 119).

$$\angle AOB_n \simeq - \angle AOA_n \simeq - n \angle AOA_1$$

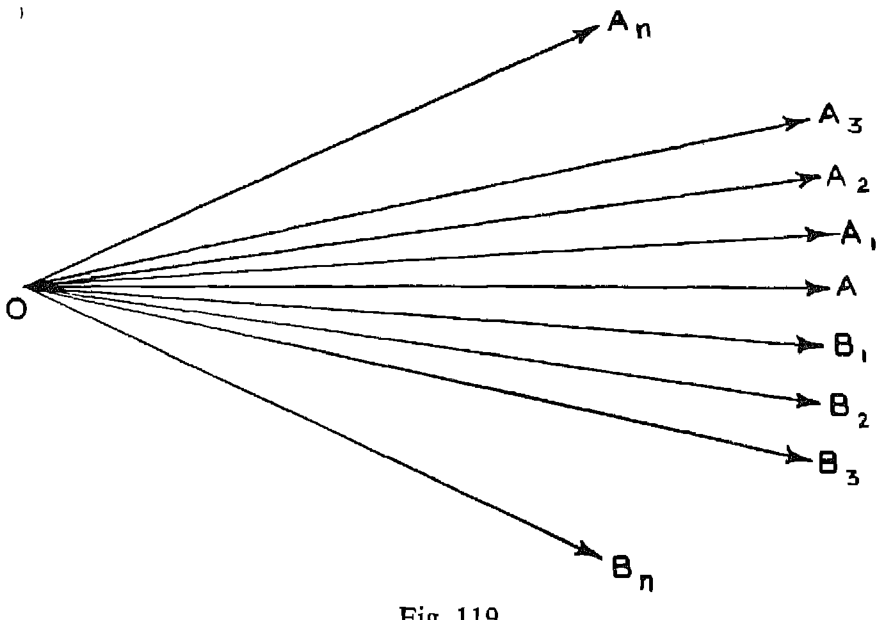


Fig 119.

EXERCISES 3.2

1. On a line 1, A, A_1, A_2 ...., A_{10}, and B_1, B_2,........, B_{10}, are points such that:

$$\overrightarrow{AA_1} \simeq \overrightarrow{A_1A_2} \simeq \overrightarrow{A_2A_3} \simeq \ldots\ldots \simeq \overrightarrow{A_9A_{10}}$$

and $\overrightarrow{AB_1} \simeq \overrightarrow{B_1B_2} \simeq \overrightarrow{B_2B_3} \simeq \cdots \simeq \overrightarrow{B_9B_{10}} \simeq -\overrightarrow{AA_1}$

Name 4 segments congruent to each of:

$$4\,\overrightarrow{AA_1},\ 7\,\overrightarrow{AA_1},\ 8\,\overrightarrow{AA_1},\ 15\,\overrightarrow{AA_1},\ 17\,\overrightarrow{AA_1}$$

$$-6\,\overrightarrow{AA_1},\ -16\,\overrightarrow{AA_1},\ -9\,\overrightarrow{AA_1},\ -14\,\overrightarrow{AA_1},\ -15\,\overrightarrow{AA_1}$$

Name all segments congruent to $2\,\overrightarrow{AA_1}$, $19\,\overrightarrow{AA_1}$, $-18\,\overrightarrow{AA_1}$, $-17\overrightarrow{AA_1}$.

How many times $\overrightarrow{AA_1}$ are the following segments:

$$\overrightarrow{A_5A_9},\ \overrightarrow{B_4A_6},\ \overrightarrow{A_3B_7},\ \overrightarrow{A_6B_8},\ \overrightarrow{B_{10}A_2},\ \overrightarrow{A_9B_2},$$

2. Through a point O, OA, OA_1, OA_2, ... , OA_{10}; OB_1, OB_2, ...,
OB_{10} are rays such that:

$$\angle AOA_1 \simeq \angle A_1OA_2 \simeq \angle A_2OA_3 \simeq \ldots\ldots \simeq \angle A_9OA_{10} \text{ and}$$
$$\angle AOB_1 \simeq \angle B_1OB_2 \simeq \angle B_2OB_3 \simeq \ldots\ldots \simeq \angle B_9OB_{10} \simeq -\angle AOA_1$$

Name 4 angles congruent to each of

$4\,\angle AOA_1$, $7\,\angle AOA_1$, $8\,\angle AOA_1$, $15\,\angle AOA_1$, $17\,\angle AOA_1$, $-6\,\angle AOA_1$, $-16\,\angle AOA_1$, $-9\,\angle AOA_1$, $-14\,\angle AOA_1$, $-15\,\angle AOA_1$.

Name all angles congruent to: $2\,\angle AOA_1$, $19\,\angle AOA_1$, $-18\,\angle AOA_1$, $-17\,\angle AOA_1$.

How many times $\angle AOA_1$ are the following angles :—

$\angle A_5OA_9$, $\angle B_4OB_6$, $\angle A_3OB_7$, $\angle A_6OB_8$, $\angle B_{10}OA_2$, $\angle A_9OB_2$.

3.5 Null Angle or Flat Angle

If the ray OB of $\angle AOB$ rotates towards OA and ultimately coincides with OA, then the angle formed by them is called a *Null angle* or a *Flat angle*, just as when B coincides with A, the segment AB is a null segment. If a null angle is added to or

subtracted from any other angle, we get the same angle, according to our definition.

$$\angle AOB - \angle AOB = \angle AOB + \angle BOA = \angle AOA = \text{a null angle.}$$

Straight Angle

If the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ occupy opposite directions on the same line as in Fig. 120, $\angle AOB$ is called a *straight angle.*

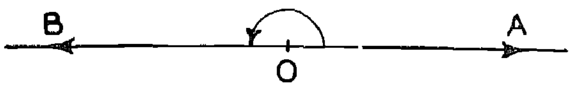


Fig. 120

A right angle

If OC is the symmetry line of the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ of *a straight angle*, then we know that $\angle AOC \simeq \angle COB$.

If CO is produced to D, then (Fig. 121), $\angle AOC \simeq \angle BOD$

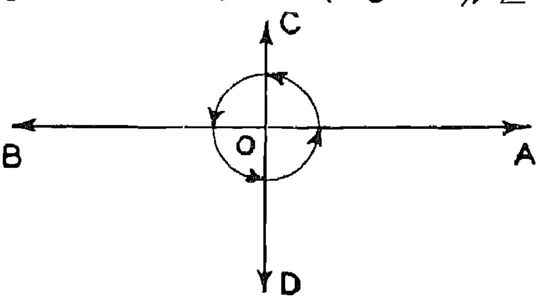


Fig. 121.

because they are vertically opposite angles, and $\angle COB \simeq \angle DOA$ because they are also vertically opposite angles.

Therefore the four angles $\angle AOC$, $\angle COB$, $\angle BOD$, $\angle DOA$ are all congruent to each other. If the paper is folded along the line COD, the ray OA falls on OB, because OC is their line of symmetry. You can verify that if the paper is folded along the line AOB, the ray OC falls on the ray OD, i.e., the two lines AOB, COD are perpendicular to each other.

The four angles formed by any two perpendicular lines are mutually congruent and each one of these is called a *right angle*;

All right angles are congruent to each other. The straight angle AOB is the sum of the two right angles AOC and COB, or a straight angle is equal to two right angles.

3.6 Comparison of Segments

We know already how to compare two segments on the same coordinate line according to their lengths. Now we will take up the precise notion of comparison of any two segments, AB and CD.

If you want to compare your pencil with that of your friend, you hold them side by side so that their blunt ends are together as shown in Fig. 122. You say that the pencil A is shorter because its sharp end falls within the sharp end of the pencil B.

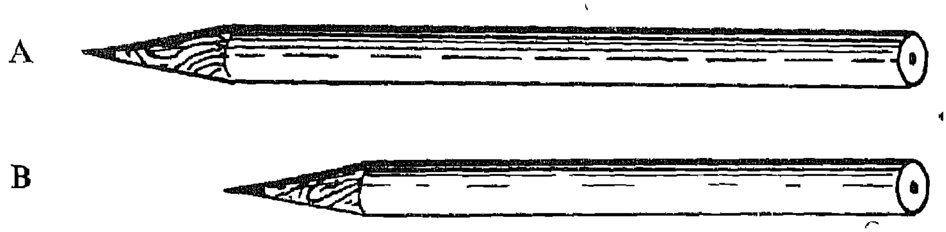


Fig. 122.

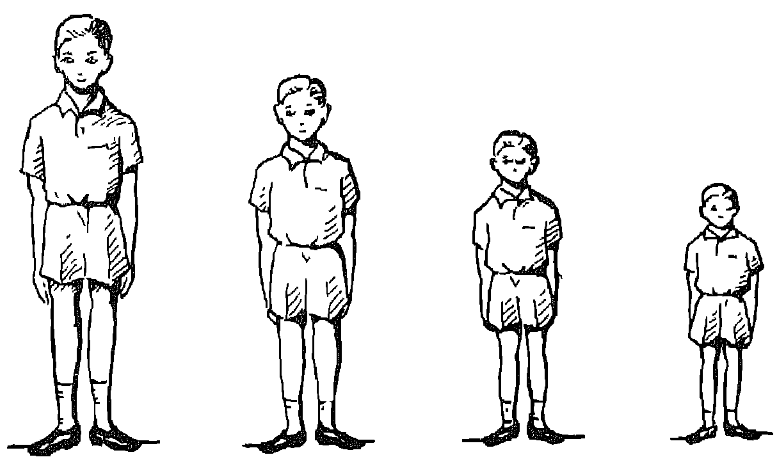


Fig. 123.

You know how to compare your heights with that of your friends in the physical training class. Your friend comes and stands by your side,

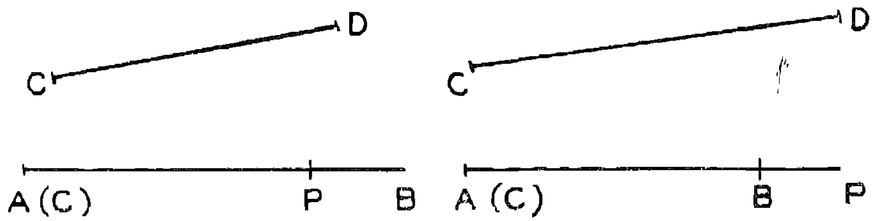
(Fig. 123), the top of his head may be either below or above yours. If the top of his head is below yours, we say *you are taller than he* or *he is shorter than you.* If the top of his head is above yours, we say *he is taller than you* or *you are shorter than he.*

Thus by comparing your heights two by two, you are made to stand in a row in the order of decreasing heights. If we look at the row from the other end, it will be in the order of increasing heights. *But though the arrangement is according to the order of height the actual height of no boy is measured.*

Similarly, if we want to compare two segments AB and CD, we map CD on the line of AB by two reflections such that C is mapped on to A and $\overrightarrow{CD}$ along the ray $\overrightarrow{AB}$ (Fig. 124).

By this D will be mapped on to B if and only if AB $\cong$ CD.

If AB is not congruent to CD, and if D is mapped on to P, P may



(a) Fig. 124. (b)

fall between A and B (i.e., P $<$ B or B $>$ P) as in Fig. 124 (a). In this case we say AB is greater than CD or CD is less than AB, and write :

$$AB > CD$$
$$\text{or} \quad CD < AB$$

But if P falls beyond B as in Fig. 124 (b), we say AB is less than CD or CD is greater than AB, and we write :

$$AB < CD$$
$$\text{or} \quad CD > AB$$

We know that if Rama is taller than Krishna and Krishna is taller than Hari, then Rama is definitely taller than Hari

i.e., $\left.\begin{array}{l}\text{Rama is taller than Krishna} \\ \text{and Krishna is taller than Hari}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\text{Rama is taller} \\ \text{than Hari}\end{array}\right.$

Similarly $\{AB > CD \text{ and } CD > EF\} \Rightarrow AB > EF$

Segments can be ordered just like numbers or like your classmates who can be arranged according to height.

3.7 Comparison of Angles

Two angles (which are less than a straight angle) can also be compared in a similar manner.

Suppose we want to compare $\angle AOB$, $\angle CO'D$. We assume they are *similarly oriented.* We make two reflections so that the ray O′C is mapped on to OA. Then by this O′D is mapped on to the ray OB if and only if $\angle AOB \simeq \angle CO'D$ If $\angle AOB$ is not congruent to $\angle CO'D$. and O′D is mapped on to the ray OP, then OP may belong either to the interior or to the exterior of $\angle AOB$ (Fig. 125).

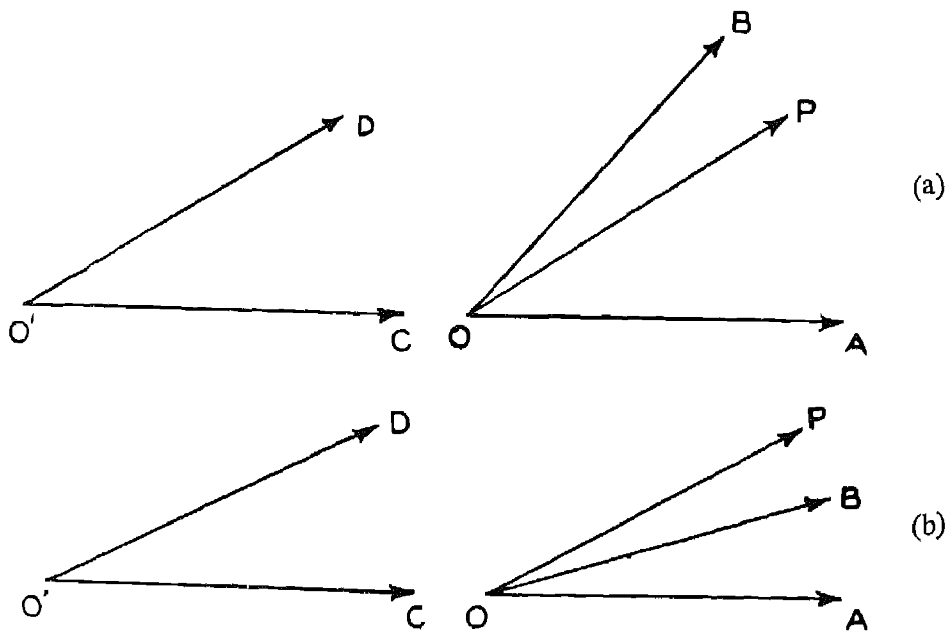


Fig. 125.

If OP belongs to the interior of $\angle AOB$, as in Fig. 125 (a), i.e., if the ray OP falls between the rays OA and OB, we say $\angle AOB$ is greater than $\angle CO'D$, or $\angle CO'D$ is less than $\angle AOB$, and write this as:

$$\angle AOB > \angle CO'D$$

or $$\angle CO'D < \angle AOB$$

But if OP belongs to the exterior of $\angle AOB$, as in Fig. 125 (b), we say $\angle AOB$ is less than $\angle CO'D$, or $\angle CO'D$ is greater than $\angle AOB$, and write this as:

$$\angle AOB < \angle CO'D$$

or $$\angle CO'D > \angle AOB$$

Also $$\left.\begin{array}{l} \angle AOB > \angle CO'D \\ \text{and } \angle CO'D > \angle EO''F \end{array}\right\} \Longrightarrow \angle AOB > \angle EO''F.$$

It is common to use the Greek letters θ (theta), Φ (phi) and ψ (psi) to denote angles. Using these we can write the above result as:

$$\{\theta > \phi \text{ and } \phi > \Psi\} \Rightarrow \theta > \Psi$$

We have defined the angle between two rays in such a way that every angle is less than a straight angle, as you can see in Fig. 126.

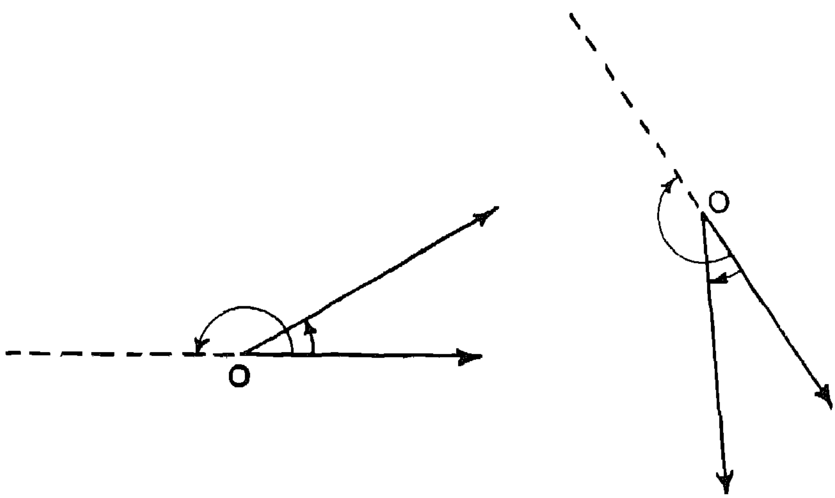


Fig. 126.

Therefore all angles less than a straight angle can be ordered just like segments.

An angle which is less than a right angle is called an *acute angle*. An angle which is greater than a right angle but less than a straight angle is called an *obtuse angle*.

3·8 Angle and Rotation

The angle between the hands of a clock will be different at different times, (Fig. 127). At 12 o' clock, they form a null angle. At 6 o' clock, they form a straight angle. At 3 o' clock and 9 o' clock, they form a right angle; but these angles from the minute hand to the hour hand are oppositely oriented. At 4 o' clock and 8 o' clock the angles are skew-congruent i. e., oppositely oriented.

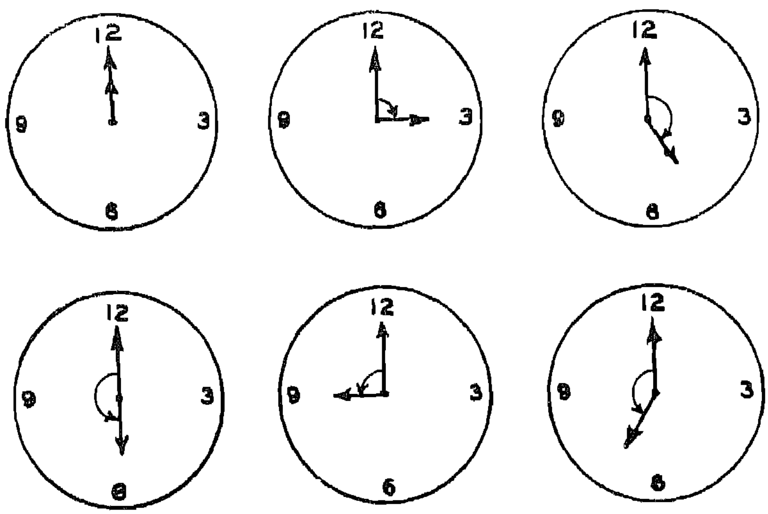


Fig. 127.

In every hour, the minute hand traces all the four right angles $\alpha, \beta, \gamma, \delta$ successively. Every 3 hours the hour hand traces a right angle (Fig. 128).

When a wheel rotates about its centre O, the spokes also rotate. The spoke OP makes different angles θ, Φ, ψ, etc., at different times, with

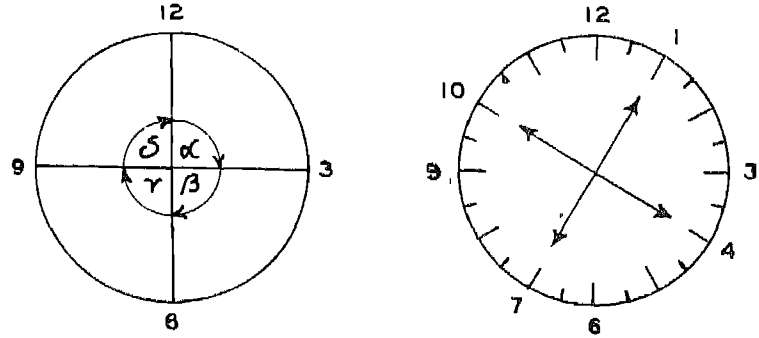


Fig. 128.

the fixed ray OA. When it makes a complete revolution, OP traces four right angles (Fig. 129).

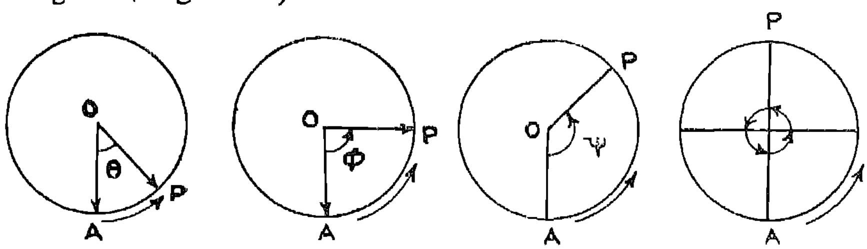


Fig. 129.

Have you observed the sun's path from morning to evening? Suppose you are at O (Fig. 130). The sun rises in the east (Note:–This happens exactly twice every year about 21st March and 21st September and on other days near to these days it rises approximately in the east) at E and goes up to the zenith at Z and comes down and sets in the west at W. EW is called the horizontal line. If S is the position of the sun, the ray OS starts from the position OE at sunrise and occupies the position OZ at noon. Therefore, in 6 hours it traces the right angle EOZ; from

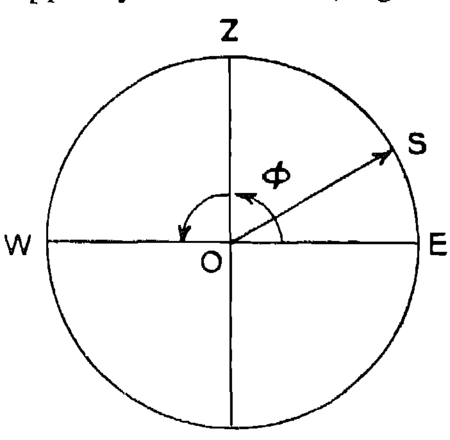


Fig. 130.

noon to sunset again it traces the right angle $\angle ZOW$. In twelve hours, the radius to the sun traces *two right angles* whereas the hour hand of a clock traces *four right angles*. In one full day, i.e., from today's sunrise to tomorrow's sunrise OS traces 4 right angles.

3.9 Measure of An Angle

The unit of measure of an angle now in use is perhaps of Babylonian origin. Geometry was used to understand the apparent motion of the sun, moon etc , which are so important in determining the seasons (summer, winter and rainy seasons).

One year is approximately 360 days (more precisely 365.2422 days). That is, the earth takes 360 days to go once completly round the sun. In other words, by the time the earth goes round the sun once it will have rotated 360 times about its own axis.

(Actually the multiple of 10 nearest to the number of days in the year is 370, but this is not as convenient a number as 360. 370 has not got as many simple divisors as 360 has viz., (2, 3, 4, 5, ... etc.,).

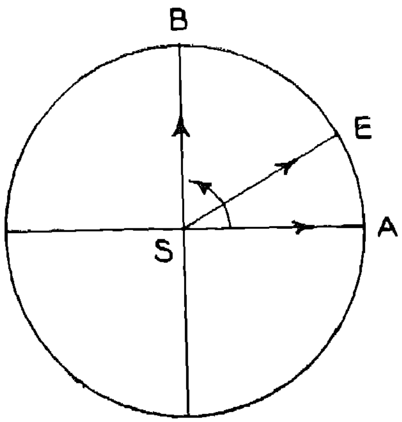


Fig. 131.

Thus if S denotes the position of the sun and E the earth, as the earth moves round the sun starting from A, it takes one year or 360 days to come back to A (Fig. 131). The angle traced by SE in one day is called one *degree* and is chosen as the unit to measure an angle. A right angle i . e , the angle traced in 90 days is made up of 90 such unit angles.

$\therefore$ One right angle = ninety degrees,
denoted by the symbol: 90°.

A straight angle is equal to 180 degrees or 180°.

The sum of four right angles is 360 degrees which is written as 360° (Fig 132).

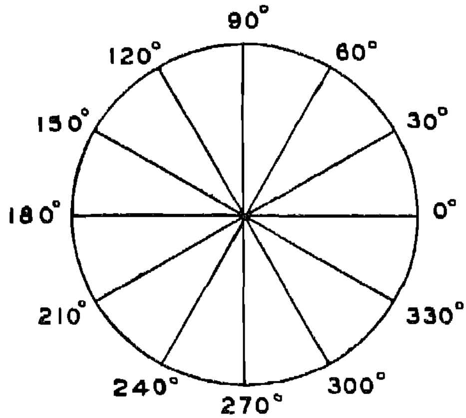


Fig. 132.

With this choice of the unit angle we define the measure of an angle as the number of unit angles of which it is made i. e., the number of degrees contained in it, just as the measure of a segment is determined in terms of the chosen unit like centimeter, meter, mile or inch.

3.10 Measure of Certain Angles

If OP is the symmetry line of a right angle AOB, then $\angle AOP \simeq \angle POB = 45°$ (Fig. 133).

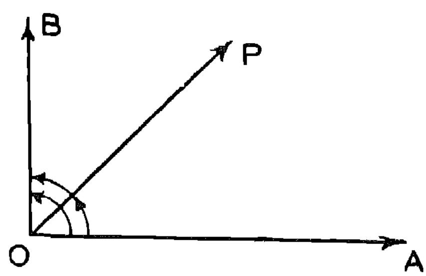


Fig. 133.

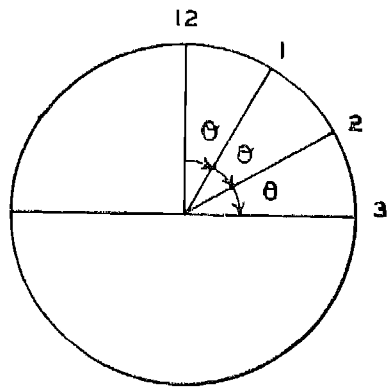


Fig 134.

i.e., $\left.\begin{array}{l}\text{Rama is taller than Krishna}\\ \text{and Krishna is taller than Hari}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\text{Rama is taller}\\ \text{than Hari}\end{array}\right.$

Similarly $\{AB > CD \text{ and } CD > EF\} \Rightarrow AB > EF$

Segments can be ordered just like numbers or like your classmates who can be arranged according to height.

3.7 Comparison of Angles

Two angles (which are less than a straight angle) can also be compared in a similar manner.

Suppose we want to compare $\angle AOB$, $\angle CO'D$. We assume they are *similarly oriented.* We make two reflections so that the ray O′C is mapped on to OA. Then by this O′D is mapped on to the ray OB if and only if $\angle AOB \simeq \angle CO'D$. If $\angle AOB$ is not congruent to $\angle CO'D$. and O′D is mapped on to the ray OP, then OP may belong either to the interior or to the exterior of $\angle AOB$ (Fig. 125).

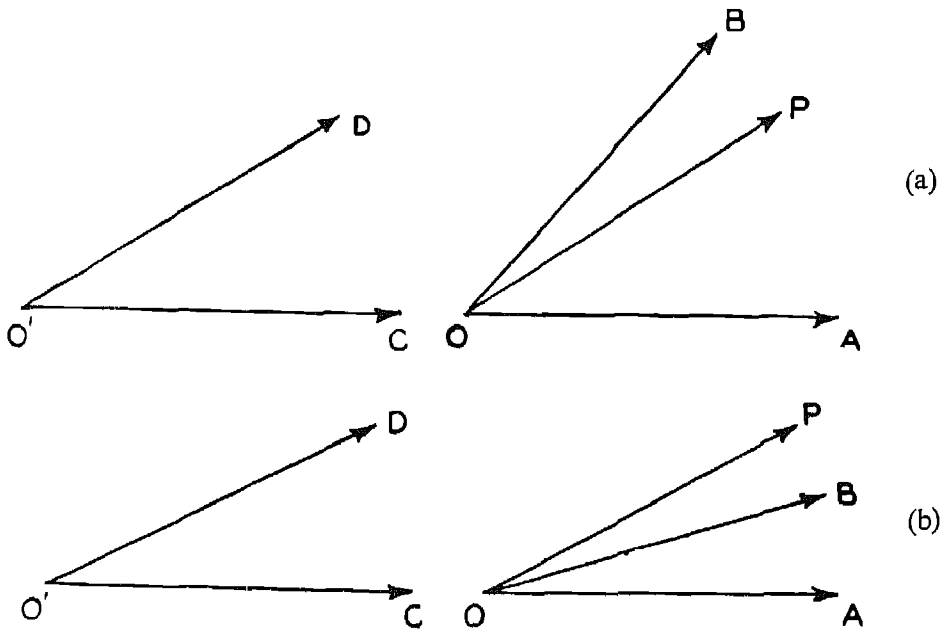


Fig. 125.

subtracted from any other angle, we get the same angle, according to our definition.

$$\angle AOB - \angle AOB = \angle AOB + \angle BOA = \angle AOA = \text{a null angle.}$$

Straight Angle

If the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ occupy opposite directions on the same line as in Fig. 120, $\angle AOB$ is called a *straight angle.*

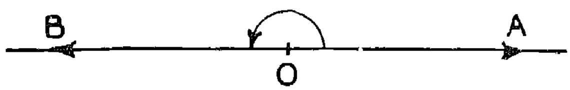


Fig. 120

A right angle

If OC is the symmetry line of the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ of *a straight angle,* then we know that $\angle AOC \simeq \angle COB$.

If CO is produced to D, then (Fig. 121), $\angle AOC \simeq \angle BOD$

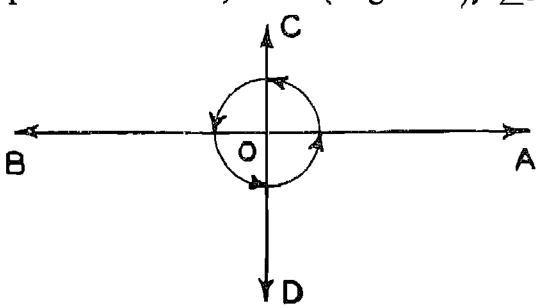


Fig. 121.

because they are vertically opposite angles, and $\angle COB \simeq \angle DOA$ because they are also vertically opposite angles.

Therefore the four angles $\angle AOC$, $\angle COB$, $\angle BOD$, $\angle DOA$ are all congruent to each other. If the paper is folded along the line COD, the ray OA falls on OB, because OC is their line of symmetry. You can verify that if the paper is folded along the line AOB, the ray OC falls on the ray OD, i.e., the two lines AOB, COD are perpendicular to each other.

The four angles formed by any two perpendicular lines are mutually congruent and each one of these is called a *right angle:*

O is the centre of the semicircle XYX′. The line XOX′ is called the base. Beginning from X and going up to X′, the semicircle XYX′ is graduated at equal intervals of arc such that the circular arc between any two consecutive markings subtends an angle equal to the unit angle i.e., 1° at O. These markings are numbered from both the ends X and X′. The numbers beginning with zero at X run along the outer semicircle up to 180 at X′ (Fig. 135). In this the number corresponding to a marking P is the number of degrees in ∠XOP. The other series of numbers begins with zero at X′ and runs along the inner semicircle, up to 180 at X. In this the number corresponding to the marking P is the number of degrees contained in ∠X′OP. Since each of ∠XOY and ∠X′OY is a right angle each contains 90°, and therefore the number corresponding to Y in each series is 90. The numbers along the outer semicircle increase in the clockwise direction and those along the inner semicircle in the opposite direction.

If you want to measure ∠ABC, place the protractor on the sheet of paper so that O coincides with B and the base falls along the ray BA, and the ray BC on the side of the protractor. Note the marking on the protractor nearest to the ray BC.

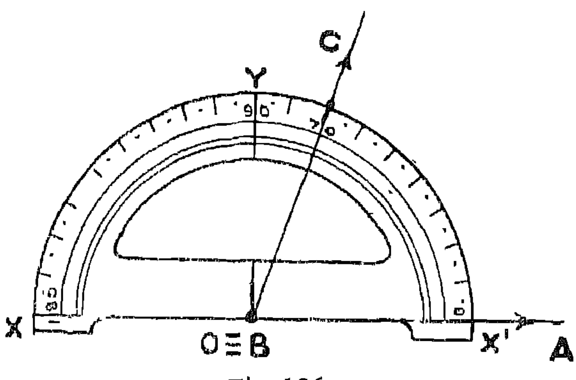


Fig. 136.

If the ray BA coincides with the ray OX′, note the number corresponding to this marking in that series which begins from zero at X′, i. e, which runs along the inner semicircle. If the ray BA coincides

with the ray OX, then note the number corresponding to this marking in that series which begins from zero at X, i. e., which runs along the outer semicircle (Fig 136). This number gives the number of degrees contained in $\angle ABC$.

If this number is 70, say, the measure of the angle is 70°. It is an oriented angle equal to 70°, if $\angle$ABC is positively oriented. And $\angle$CBA is negatively oriented and is equal to – 70°. If you find it more convenient to place the protractor so that the base line falls along the line of the ray BC, place it so that O coincides with B and X′OX falls along the line of the ray BC.

In this case note the number, as before, corresponding to the marking on the protractor nearest to the ray BA. Both these methods give the same measure of the angle.

The rays of the angle must be drawn long enough to take the reading with the protractor conveniently. You must read very carefully the correct number at the required marking. Take a protractor and examine the pairs of numbers at various markings. You will find that if one number is greater than 90 the other is less than 90. (In fact the sum at any marking is 180). After some practice by just looking at an angle you will be able to say easily whether it is an *acute angle* i.e., less than 90°, or an *obtuse angle* i.e., greater than 90°, unless it is very near 90°, say, between 80° and 100°. If it is an acute angle, i.e., less than a right angle, read that number at the marking which is less than 90. If it is an obtuse angle i.e., greater than a right angle read that number at the marking which is greater than 90, as the measure of the angle. Thus you can easily choose the correct number. Your ability to measure an angle correctly and quickly with a protractor is a proof of your intelligence.

It is obvious that the measure of all congruent angles is the same. If the measure of $\angle$ABC is x and that of $\angle$PQR is y,

then the measure of $\angle ABC + \angle PQR = x + y$
and the measure of $\angle ABC - \angle PQR = x - y$.
The measure of $n.\ \angle ABC = nx$.

noon to sunset again it traces the right angle $\angle ZOW$. In twelve hours, the radius to the sun traces *two right angles* whereas the hour hand of a clock traces *four right angles*. In one full day, i.e., from today's sunrise to tomorrow's sunrise OS traces 4 right angles.

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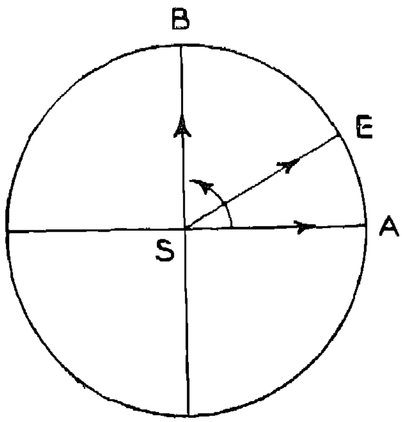


Fig. 131.

Thus if S denotes the position of the sun and E the earth, as the earth moves round the sun starting from A, it takes one year or 360 days to come back to A (Fig. 131). The angle traced by SE in one day is called one *degree* and is chosen as the unit to measure an angle. A right angle i.e., the angle traced in 90 days is made up of 90 such unit angles.

$\therefore$ One right angle = ninety degrees,
denoted by the symbol: $90°$.

If OP belongs to the interior of $\angle AOB$, as in Fig. 125 (a), i.e., if the ray OP falls between the rays OA and OB, we say $\angle AOB$ is greater than $\angle CO'D$, or $\angle CO'D$ is less than $\angle AOB$, and write this as:

$$\angle AOB > \angle CO'D$$

or

$$\angle CO'D < \angle AOB$$

But if OP belongs to the exterior of $\angle AOB$, as in Fig. 125 (b), we say $\angle AOB$ is less than $\angle CO'D$, or $\angle CO'D$ is greater than $\angle AOB$, and write this as:

$$\angle AOB < \angle CO'D$$

or

$$\angle CO'D > \angle AOB$$

Also

$$\left.\begin{array}{l} \angle AOB > \angle CO'D \\ \text{and } \angle CO'D > \angle EO''F \end{array}\right\} \Longrightarrow \angle AOB > \angle EO''F.$$

It is common to use the Greek letters θ (theta), Φ (phi) and ψ (psi) to denote angles. Using these we can write the above result as:

$$\{\theta > \phi \text{ and } \phi > \psi\} \Rightarrow \theta > \psi$$

We have defined the angle between two rays in such a way that every angle is less than a straight angle, as you can see in Fig. 126.

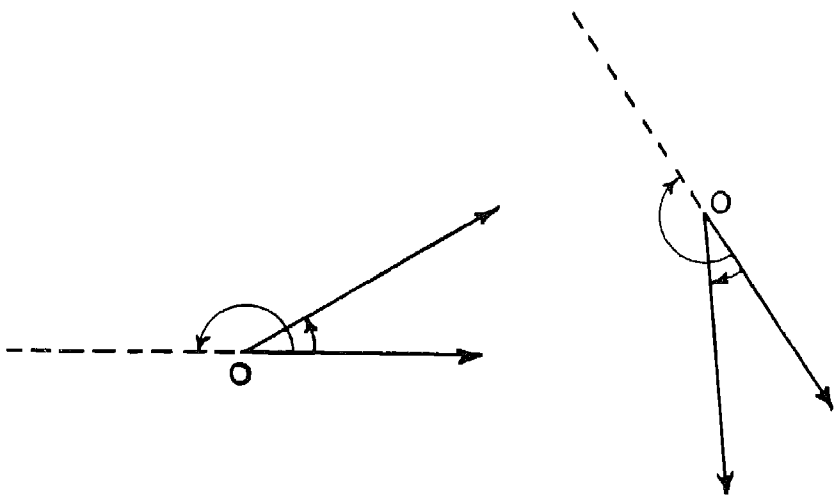


Fig. 126.

and B is a point on the other, measure $\angle$OAB and $\angle$ABO and find their sum.

3) Draw any three different triangles and measure their angles. Find the sum of the three angles of each triangle.

4) In Fig. 140, angles *1*, *2*, *3*, *4*, *5*, and *6* are all congruent and $\angle$AOB is a straight angle. What is the measure of each of them in degrees and in radians ?

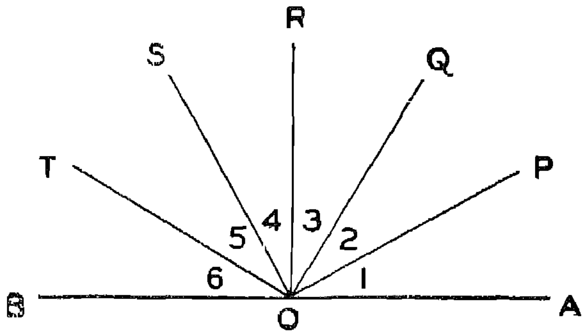


Fig. 140.

What is the measure of each of the following angles in degrees and in radians, taking APQ............B as the positive orientation ?

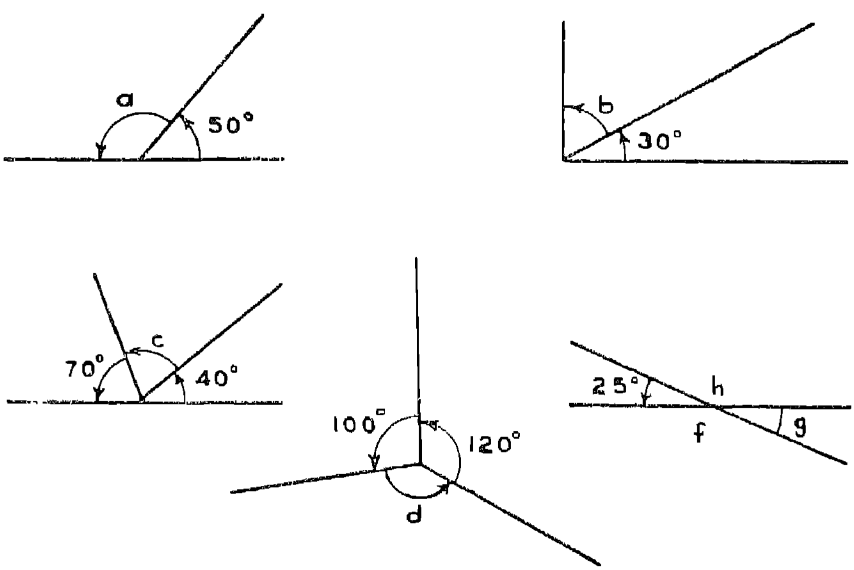


Fig. 141.

Also in Fig. 140, what are the measures of the angles :

(1) $\angle AOQ$ (2) $\angle SOQ$ (3) $\angle ROA$ (4) $\angle POB$
(5) $\angle BOA$ (6) $\angle TOP$ (7) $\angle AOT$ (8) $\angle TOS$.

Verify your answers by measurement.

5) Draw two figures each of :

(a) an acute angle,
(b) an obtuse angle.

Estimate their size in terms of a right angle.

6) Use your protractor to draw pairs of lines at angles of :

(a) 20° (b) 30° (c) 100° (d) 130° (e) 70° (f) 110°.

7) What is the angle traced by the minute hand of a clock in 5 minutes ?

8) The speedometer needle in a car turns clockwise through a right angle when the speed of the car increases from 20 to 50 km. p. h Through what angle and in what direction will the needle turn, when the car changes from :

(a) 0 to 10 km.p.h (b) 10 to 30 km.p.h
(c) 70 to 40 km.p.h (d) 35 to 25 km.p.h
(e) 24 to 29 km.p.h (f) 48 to 47 km.p.h

9) Construct the isosceles triangle ABC given :

(1) $|AB| = |AC| = 7$ cm. $|BC| = 5$ cm
(2) $|AB| = |AC| = 5$ cm. $|BC| = 8$ cm.

Measure in each case angles B and C.

10) Construct a square by paper folding. Measure its angles. Fold along the diagonals and find the angles between them by measurement.

11) Draw a circle with centre O. Mark any three points A,B and C, on it. Measure $\angle ABC$, $\angle AOC$; $\angle BCA$, $\angle BOA$; $\angle CAB$, $\angle COB$, What rule do you observe ?

12) Draw a diameter PQ of a circle. Mark any other four points A,B,C,D on the circle. Measure the angles $\angle PAQ$, $\angle PBQ$, $\angle PCQ$ and $\angle PDQ$.

What can you conclude ?

13) Construct any two intersecting lines and the bisectors of the angles formed by them by paper folding. Measure the angles between these bisectors. Show that they are at right angles. Verify this by paper folding.

14) If one angle of a linear pair is 49°, what is the other angle ?

15) Two lines intersect ; one of the angles formed is 60°, what are the other angles ?

16) Can the sum of 3 angles formed by two intersecting lines be equal to 175° ? Why ?

17) By paper folding construct an isosceles triangle, i.e., a triangle with 2 congruent sides. Fold along the symmetry line of the congruent sides. Measure the angle between this line and the base of the triangle.

Verify the above measurement by paper folding.

18) By paper folding construct a triangle ABC with $\angle ABC = 90°$. Fold along the line through B perpendicular to AC.
Measure the angles of the triangles so formed and those of triangle ABC.

19) By paper folding construct any triangle ABC. Mark the midpoints X and Y of AB and AC. Fold along the line XY. Measure $\angle AXY$, $\angle ABC$, $\angle AYX$, $\angle ACB$, segments XY and BC. Repeat the above for two more triangles What property can you guess about the segments XY and BC ?

20) Make a sketch of your route to school showing where you change your direction and approximately the angle through which you turn.

21) Construct 4 different rectangles of side lengths.

(1) 8 cm, 5 cm,
(2) 7 cm, 6 cm,
(3) 9 cm, 3 cm,
(4) 6 cm, 4 cm.

Measure their diagonals. What do you learn ?

4

Constructions

4.1 To Draw the Symmetry Line of a Pair of Points Using the Ruler and the Compasses.

Let A, B be the given points (Fig. 142). It is required to bisect the segment AB. With A as centre and a convenient radius (this should be greater than half the distance between A and B) draw an arc. With B as centre, and the same radius, draw another arc, so as to cut the former arc at the points C and D. It will be found that C and D lie on opposite sides of the line AB. Join CD, and let CD cut AB at E. Then E will be the middle point of AB, CD will be the symmetry line of A and B.

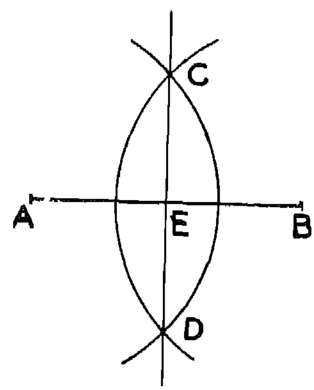


Fig. 142.

(1) Verify by means of a pair of dividers that AE $\simeq$ EB.

(2) Verify by folding successively about AB and then about CD, that CD is the symmetry line of AB and that AB is the symmetry line of CD. Thus AB $\perp$ CD.

(3) Verify by means of dividers that CE $\cong$ ED.

Definition:

CD is called the *perpendicular bisector of AB*, or the *right bisector of AB*.

EXERCISE 4 1

1. Draw a line segment of length 6.3 cm. and construct its line of symmetry.
2. Draw a segment AB of length 2.8 inches, and bisect it perpendicularly at C. Bisect AC at D. Bisect AD at E. Measure AC, AD, AE.

4.2 To Construct an Angle Congruent to a Given Angle

Let $\angle XOY$ be the given angle, and $O'X'$ a given ray. It is required to construct an angle congruent to $\angle XOY$ at O' with $O'X'$ as an arm (Fig. 143).

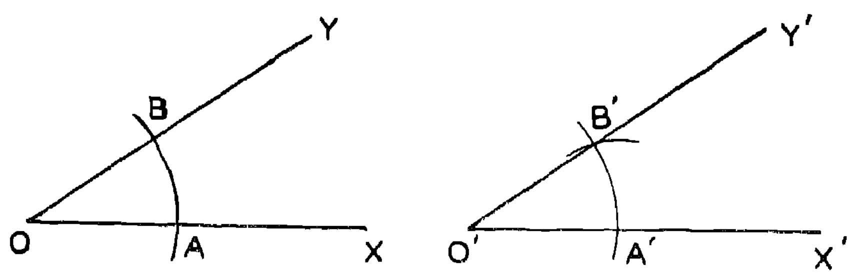


Fig. 143.

With O as centre draw an arc cutting the arms OX and OY at A and B respectively. With the same radius, draw an arc with centre O', cutting $O'X'$ at A'. Take the distance AB as radius in the compasses and with A' as centre draw an arc cutting the previous arc at B' so that $\angle XOY$ and $\angle A'O'B'$ have the same orientation. Join

O'B'. Then $\angle A'O'B'$ is the required angle. Verify by measurement, using a protractor that :

$$\angle A'O'B' \simeq \angle AOB$$

EXERCISE 4.2

1. Construct angles congruent to the angles given in Figure. 144.

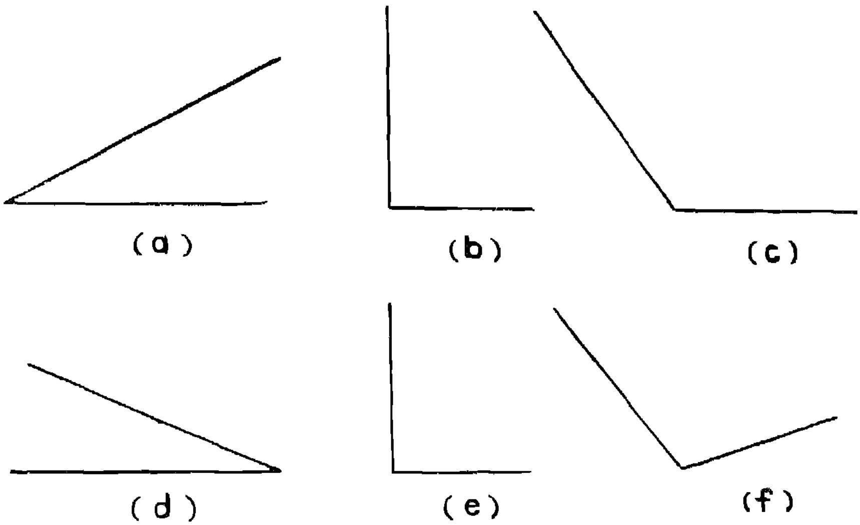


Fig. 144.

4.3 To Bisect a Given Angle i.e., to Draw the Symmetry Line of a Given Angle.

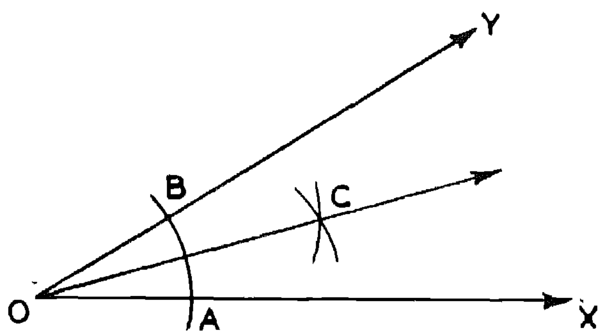


Fig. 145.

Let XOY be the given angle (Fig. 145). With O as centre and a convenient radius draw an arc to cut OX and OY at A and B respectively. With A as centre, and some convenient radius draw an arc. With B as centre and the same radius draw another arc so as to cut the former arc at C. Join OC. Then OC is the symmetry line *or* the *bisector* of $\angle$XOY.

EXTENSION

We can further bisect each of the angles AOC and COB. Each of these parts will be congruent to 1/4 of the given angle $\hat{AOB}$. By further bisection of each of these quarters, we can divide the angle AOE into 8 mutually congruent parts, next into 16 mutually congruent parts, etc.

EXERCISE 4 3

1. Verify by measurement that angles XOC and COY are congruent (Fig. 145).
2. Verify by paper folding that $\angle$COY is congruent to $\angle$XOC (Fig 145).
3. If the line OC cuts the arc AB at D, verify both by measurement and by means of a pair of dividers, that the segments AD and DB are congruent (Fig. 145).
4. Draw an angle of 64° and divide it into 4 congruent parts. Measure each angle.
5. Draw an angle of 160° and divide it into 8 congruent parts Measure each part.

4.4 To Draw a Line through a Given Point Parallel to a Given Line

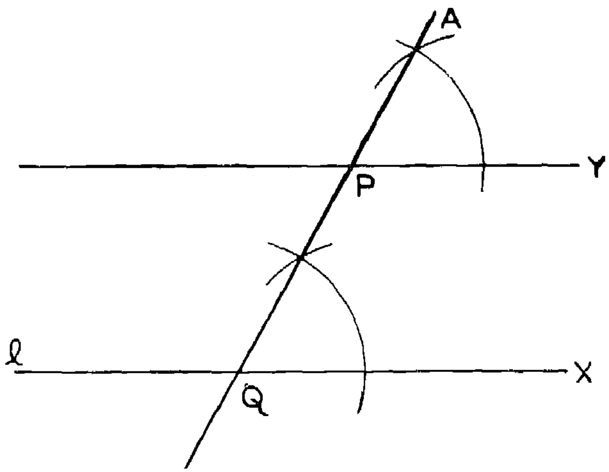


Fig. 146.

Let l be the given line and P the given point (Fig. 146). It is required to draw through P a line parallel to l.

Construction

Take any line AP cutting l at Q. At P construct $\angle$APY congruent to $\angle$AQX. Then PY will be parallel to QX.

Note :—APQ is a transversal for the parallel lines PY and QX. Here angles $\angle$XQP and $\angle$YPA are called corresponding angles or alternate angles according as A does or does not lie between P and Q. This construction makes use of the fact that if a transversal cuts a system of parallel lines, the corresponding angles are congruent and the alternate angles are congruent.

4.5 To Divide a Segment into Any Number of Congruent Parts :

Let AB be the given segment and let it be required to divid it into 5 congruent parts.

One method is given in article 2.7. Here is the second method.

Construction

Take any ray AX through A and using the ruler and compasses draw the ray BY // AX (Fig. 147) so that X and Y belong to the opposite sides of the line of the segment AB.

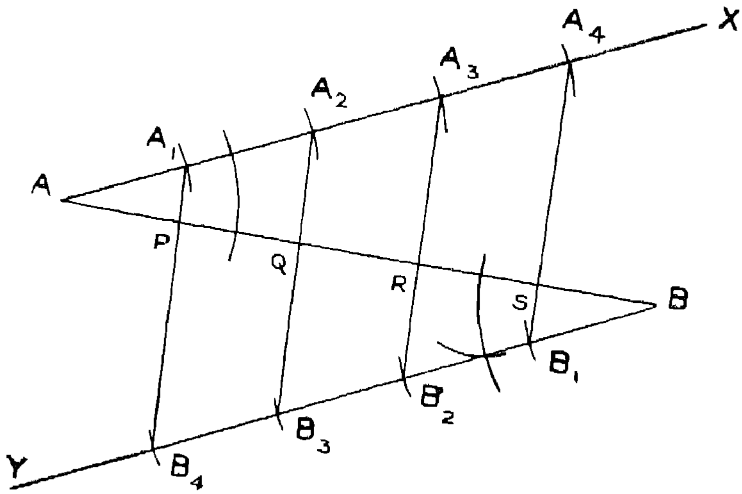


Fig. 147.

Taking any convenient length in the compasses step off four segments: $AA_1 \simeq A_1A_2 \simeq A_2A_3 \simeq A_3A_4$, on AX. With the same length in the compasses, step off four segments: $BB_1 \simeq B_1B_2 \simeq B_2B_3 \simeq B_3B_4$ along BY.

$$\text{Let } A_1B_4 \cap AB = P$$
$$A_2B_3 \cap AB = Q$$
$$A_3B_2 \cap AB = R$$
$$A_4B_1 \cap AB = S$$

Then, it will be found that $AP \simeq PQ \simeq QR \simeq RS \simeq SB \simeq 1/5\ AB$.

EXERCISE 4.4

1. Verify the result of article 4.5 by measurement and also by paper folding.
2. Check the congruence of the segments AP, PQ, QR, RS and SB using a pair of dividers (Fig. 147).

3. Measure the segments A_1B_4, A_2B_3, A_3B_2 and A_4B_1, and verify that they are mutually congruent (Fig. 147).

4 Check the congruence of these segments by means of the dividers.

5. Verify, by using a protractor, that in Fig 147, the four angles $\angle A_1PA$, $\angle A_2QA$, $\angle A_3RA$, and $\angle A_4SA$, are mutually congruent.

4.6 Construction of Certain Angles using the Ruler and the Compasses only.

Angles of 60°, 90°, 45° etc., can be easily constructed using the straight edge and compasses only, as explained below.

Angle of 60°

To construct an angle of 60° at a point O :

Take a ray OX from O, and with O as centre and any convenient length as radius draw an arc cutting OX in A (Fig. 148).

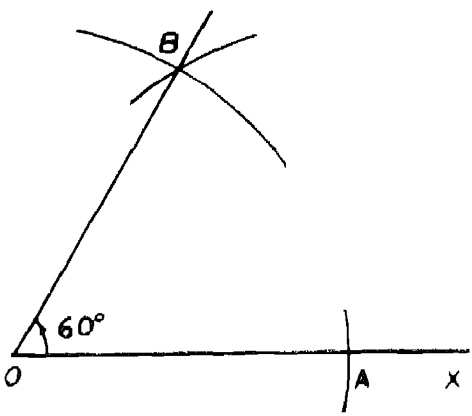


Fig. 148.

With the same radius, and A as centre draw an arc cutting the former arc at B. Join OB. On measuring the angle XOB, it will be found to be an angle of 60°.

EXERCISE 4.5

1. Bisect angle XOB (Fig. 148), and construct thereby an angle of 30°, and by further bisection obtain an angle of 15°.

2. By adding an angle of 15° to one of 30°, obtain an angle of 45°.

3. By doubling an angle of 45°, construct by ruler and compasses an angle of 90°.

4. By the use of the ruler and compasses, construct parallelogram SPQR where $|SP| = 3$cm. $|SR| = 4$cm. $\angle PSR = \frac{\pi}{4}$; measure PR and QS, QR and QP (correct to a millimeter) If PR and QS meet at O, verify by measurement that $PO \simeq OR$ and $SO \simeq OQ$.

5. Construct the parallelogram in the preceding question with

$$|PS| = |SR| = 4 \text{ cm. and } \angle PSR = \frac{\pi}{3}$$

Measure QS and PR and verify that $PR \perp QS$.

6. Draw a line l. On l mark points A, A_1, A_2, A_3, and A_4 such that

$$\begin{aligned} |AA_1| &= 1 \text{ cm}, \\ |A_1A_2| &= 2 \text{ cm}, \\ |A_2A_3| &= 4 \text{ cm}, \\ |A_3A_4| &= 3 \text{ cm}, \end{aligned}$$

Draw any other line AX. Draw A_1X_1, A_2X_2, A_3X_3, A_4X_4, each parallel to AX. Let another line m intersect AX at B, A_1X_1 at B_1, A_2X_2 at B_2, A_3X_3 at B_3 and A_4X_4 at B_4. Measure BB_1, B_1B_2, B_2B_3 and B_3B_4. What do you observe ?

4.7 Angle of 90° (or a Right Angle)

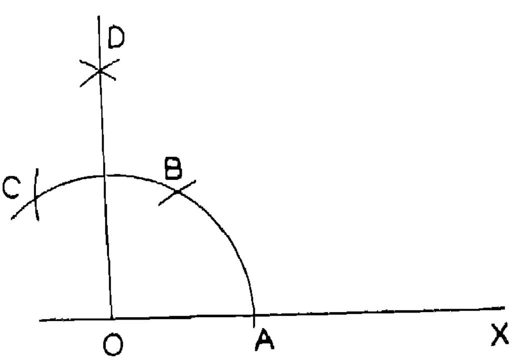


Fig 149.

A right angle may be easily constructed.

Let it be required to construct a right angle XOD, (Fig 149) at O on the ray OX without producing XO. With any convenient length as radius and O as centre draw a circle cutting OX at A. With the same length as radius cut off on this circle arcs AB and BC such that chord AB $\simeq$ chord BC $\simeq$ OA.

$$\text{Then} \quad \angle AOB = 60°, \ \angle BOC = 60°$$
$$\therefore \quad \angle AOC = 120°.$$

Now bisect $\angle BOC$ by OD (by drawing arcs with B and C and centres and a common radius to cut each other at D).

$$\text{Then} \quad \angle BOD = 30°.$$
$$\therefore \quad \angle AOD = \angle AOB + \angle BOD = 60° + 30° = 90°.$$

OD is now $\perp$ to OX. This construction therefore helps us to erect a perpendicular to a given ray at its initial point without producing the ray. (Note that in this construction only one radius can be used to draw the arcs).

4.8 To Draw a Perpendicular from Any Point to Any Straight Line

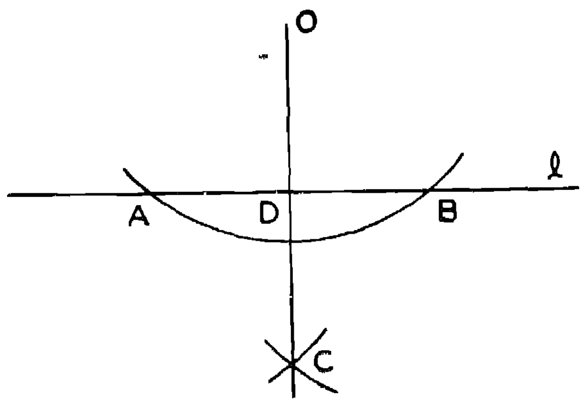


Fig. 150.

Let it be required to draw a perpendicular from a point O to a straight line l (Fig. 150). (The point O may or may not be on l). With O as centre, and a convenient radius draw an arc so as to cut l at the two points A and B.

With A and B as centres and any common radius draw two arcs to cut each other at C. Join CO, cutting AB at D. Then OD $\perp$ l. This may be verified by measurement.

When O is not on l, CO is the symmetry line of the angle $\angle AOB$ and A and B are the reflections of each other in the line OC.

EXERCISE 4.6

1. Given a line l and a point O on it, draw through O a perpendicular to l. Bisect the right angle so formed and thus construct an angle of 45°. Verify the construction by measurement.
2. Construct $\angle XOY$ equal to 60°, take P on the symmetry line of this angle at a distance of 5 cm. from O, and draw PM and PN $\perp$ to the arms of the angle, and measure them. Also measure OM, ON.
3. Construct a rectangle with adjacent sides of measure 6 cm and 8 cm. Measure its diagonals

4.9 Inscription of Regular Figures in a Given Circle

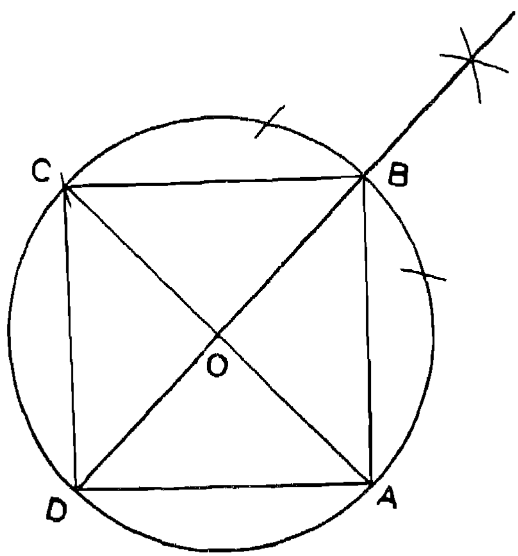


Fig 151.

1) A Square

Given a circle with centre at O, and a point A on it, let it be required to inscribe a square in the circle, having A for a vertex.

Construction :

Draw the diameter AOC through A (Fig. 151) and erect the perpendicular to AC through the centre O. This can be done by using the ruler and the compasses only (§ 4.8), as shown in the figure. Let this perpendicular cut the circle at B and D. Then BOD becomes the diameter perpendicular to the diameter AOC. Now join A, B, C, D in order Then ABCD is the square inscribed in the given circle with one vertex at A.

Measure the sides AB, BC, CD and DA and the diagonals AC and BD.

Measure the interior angles at A, B, C and D. What is their sum? Measure the angles AOB, BOC, COD and DOA.

4.10 A Regular Octagon

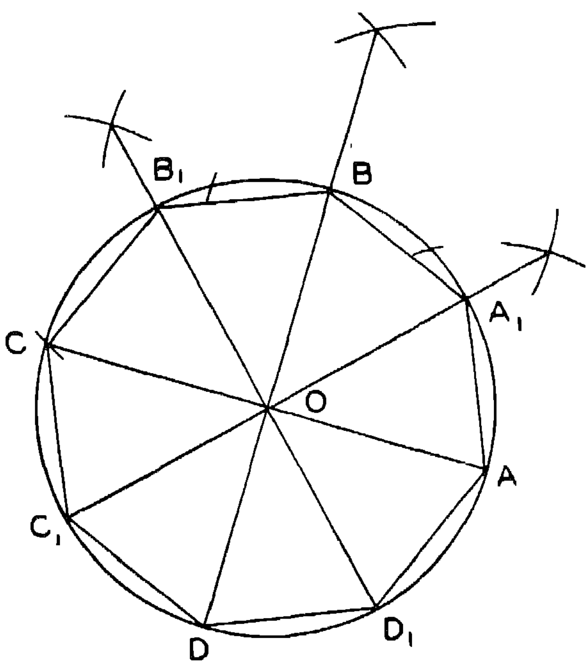


Fig. 152.

Given a circle with centre O, and a point A on it, let it be required to inscribe a regular octagon in the circle, with one vertex at A.

Construction :

First we draw the diameter AOC through A and construct the diameter BOD $\perp$ AOC.

Draw the diameter A_1C_1 bisecting the angle AOB, and the diameter $B_1D_1 \perp A_1C_1$, as shown in Fig. 152. Join A,A_1, B,B_1, C,C_1, D,D_1, A in order. Then we get the regular octagon $AA_1BB_1CC_1DD_1$ inscribed in the circle and having A for a vertex.

Measure the sides of the octagon. Measure $\angle AA_1B$ and $\angle AOA_1$. Find the sum of all the interior angles of the octagon, and the sum of all the angles at O.

4.11 A Regular Hexagon

Given a circle with centre O, and a point A on it, let it be required to inscribe a regular hexagon in the circle, with one of its vertices at A (Fig 153) .

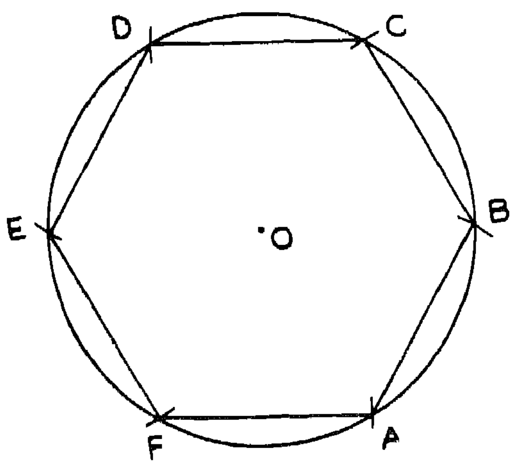


Fig. 153.

With a length OA, the radius of the circle, in the compasses, starting from A, step off congruent arcs AB, BC, CD, DE, EF along the circle. The next step comes back to A. (This is because each

angle such as $\hat{AOB}$ measures 60° and the sum of all the angles that will be formed at O equals 4 right angles or 360°. Therefore, you get precisely 6 congruent arcs such as AB). By joining A, B, C, D, E, F, A, in order, you get the regular hexagon ABCDEF inscribed in the circle, with A as a vertex. Measure the internal angles and find their sum

4.12 An Equilateral Triangle

Given a circle with centre O, let it be required to inscribe in it an equilateral triangle, with one of its vertices at a given point A on the circle (Fig. 154).

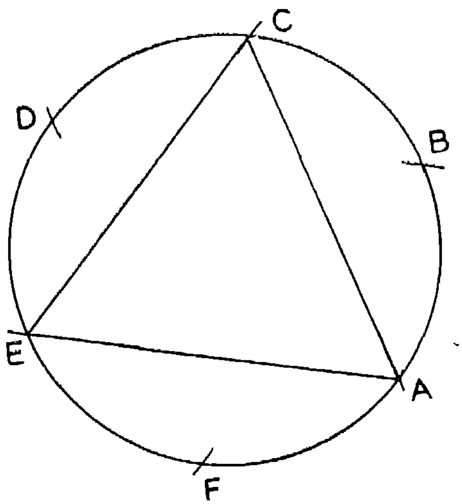


Fig. 154.

With OA as radius, and starting from A, step off congruent arcs AB, BC, CD, DE, EF, along the circle, as it was done in the previous case of a hexagon.

Join AC, CE and EA. Then ACE is the required equilateral triangle.

Measure the sides and angles of this triangle. Join AO, CO and EO and measure the angles $\angle AOC$, $\angle COE$ and $\angle EOA$.

4.13 The Six-Petalled Rose Inscribed in a Circle :

Given a circle of radius r by drawing circular arcs of radius r with centres A, B, C, D, E, F, of Fig. 153 and terminated

within the circle, we get a flower design with six petals, inscribed in the given circle (Fig. 155).

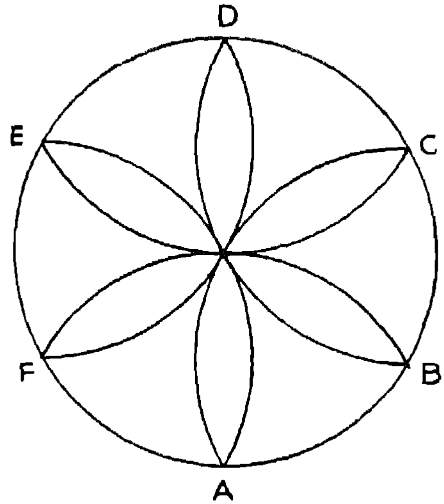


Fig. 155.

EXERCISE 4.7

Draw circles of radii 5 cm., 8 cm., 2.5″ and 3″, and in each of them inscribe the following regular figures:

(1) a square, (2) a regular octagon, (3) a regular hexagon, (4) an equilateral triangle and (5) a six-petalled rose.

Measure the sides and angles of the regular figures thus constructed.

EXERCISE 4.8

(Some interesting problems)

1. ABC is a triangle in which $\angle ABC = 30°$ and $\angle ACB = 20°$. D is a point on BC, such that $|BD| = |AC|$. Measure all the angles marked "?" in figure 156.

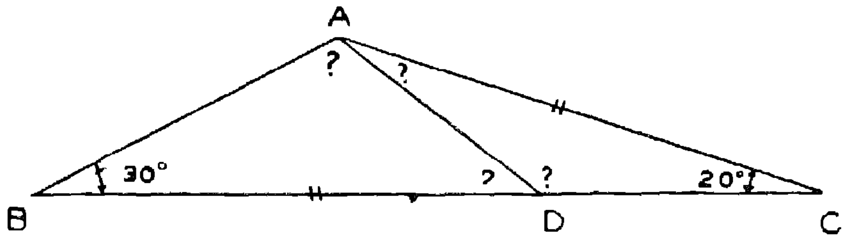


Fig. 156.

2. ABC is a triangle in which

$\angle ABC = 30°$, $\angle ACB = 40°$, D is a point on BC such that: $|BD| = |AC|$.

Measure all the angles in Fig. 157 marked "?".

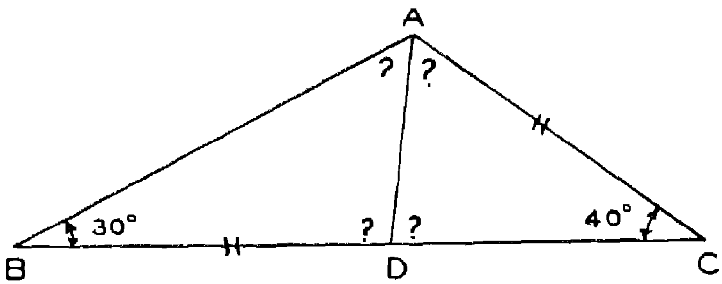


Fig. 157

3. ABC is a triangle and O is a point inside it such that $\angle OBC = \angle OCA = \angle OAB$. Calculate the angles $\angle BOC$, $\angle COA$

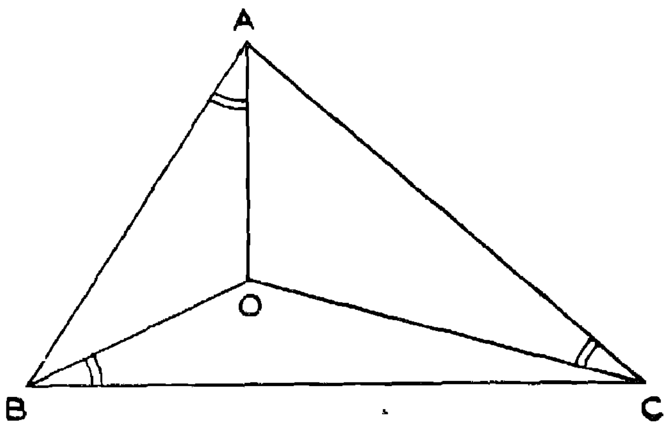


Fig. 158

and $\angle AOB$ in terms of $\angle A$, $\angle B$ and $\angle C$ (Fig. 158). (Hint: Sum of the three angles of any triangle is equal to 180°).

4 ABC is a triangle in which $\angle A=75°$, $\angle B=60°$ and $\angle C=45°$ (Fig. 159).

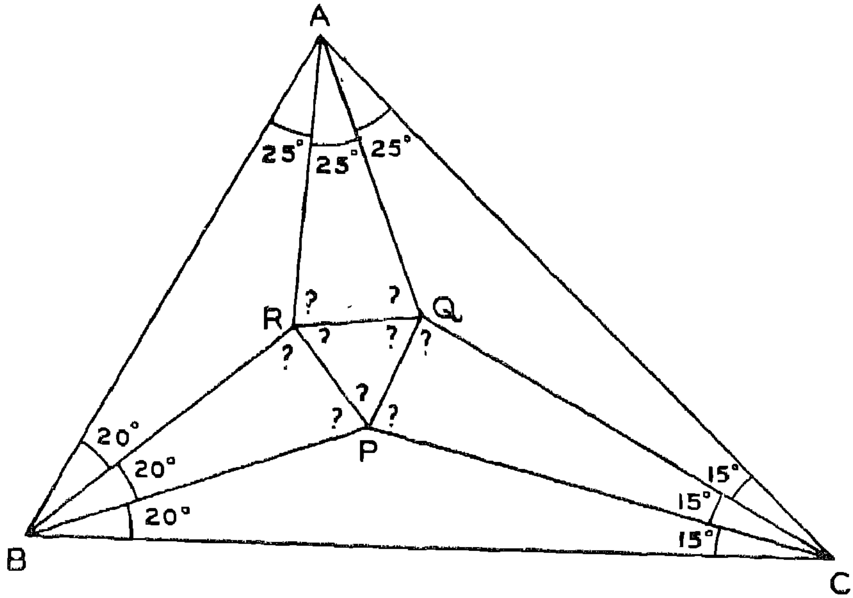


Fig. 159

The trisectors of the angles meet in P,Q,R. Measure all the angles shown in the figure.

4.14 Construction of Triangles

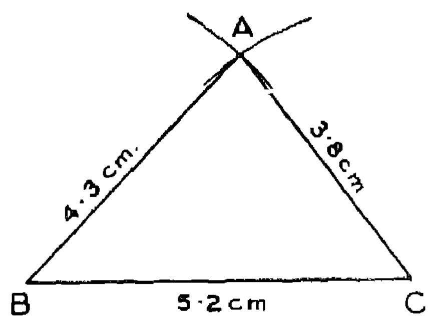


Fig. 160.

Case (1) : To construct the triangle, given its sides :

Let it be required to construct the triangle ABC given that $|AB| = 4.5$ cm. $|BC| = 5.2$ cm. and $|CA| = 3.8$ cm.

Construction :

Take the base $|BC| = 5.2$ cm. (Fig. 160). With B as centre and radius equal to 4.3 cm. draw an arc. With C as centre and 3.8 cm. as radius draw another arc to cut the former arc at A. Join AB and AC.

Then ABC is the required triangle. (Actually the two circular arcs meet at two points, say, A_1, A_2. Then both the triangle A_1BC and A_2BC satisfy the given conditions. These two triangles are reflections of each other in the line BC).

Measurements :

Measure the angles A, B and C, and find their sum.

Notation :

The lengths of the sides BC, CA and AB, which are respectively opposite to the vertices A, B and C are denoted by the corresponding small letters a, b and c. Thus in the above triangle $a = 5{\cdot}2$ cm , $b =$ 3.8 cm. and $c = 4.3$ cm.

EXERCISE 4.9

Construct the triangle ABC given that ·

(a) $a = 3''$, $b = 4''$, $c = 5''$, Measure $\angle B$ and $\angle A$.

(b) $a = 3.9$ cm, $b = 5.8$ cm, $c = 4.5$ cm.
Measure the angles.

(c) $a = 5$ cm, $b = 13$ cm, and $c = 12$ cm. Find the sum of the angles A and C.

4.15 Case 2 :

To construct a triangle, given two of its sides and the included angle.

Let it be required to construct the triangle A B C given that $c = 5.3$ cm, $b = 6.4$ cm, and A $= 43^\circ$.

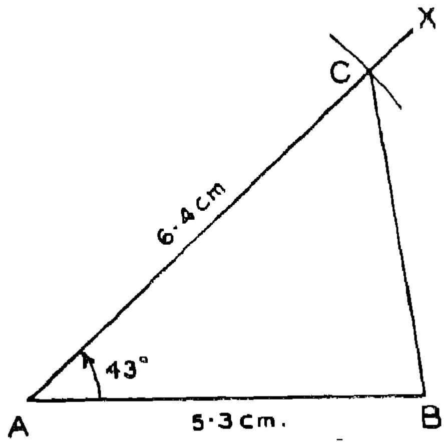


Fig. 161.

Construction :

Draw a segment AB of measure 5.3 cm (Fig.161), and construct angle BAX $= 43^\circ$, using the protractor. Take AC = 5.1 cm. along AX. Join BC. Clearly ABC is the required triangle.

EXERCISE 4.10

Construct triangle ABC given that:

(a) $c = 4.8$ cm , B $= 56^\circ$, $a = 6.3$ cm.
(b) $a = 2.5$ cm , C $= 75^\circ$, $b = 3.1$ cm.
(c) $b = 5\ 6$ cm , A $= 105^\circ$, $c = 4\ 8$ cm.

Measure the other side, and the remaining angles in each case.

Find the sum of the 3 angles A, B, C in each case.

4.16 *Case 3 :*

Given two angles and the included side, to construct the triangle.

Let it be required to construct the triangle ABC in which $a = 6.2$ cm, B = 45° and C = 65°.

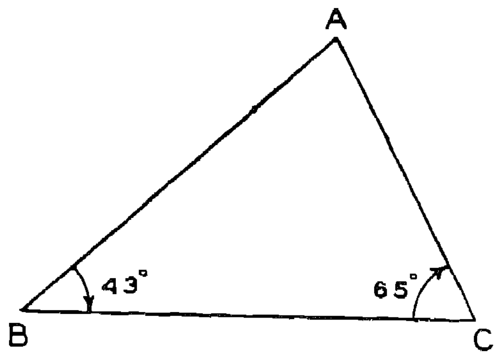


Fig. 162.

Construction :

Draw a segment BC of measure 6.2 cm. and construct $\angle$ABC = 45° and angle $\angle$BCA = 65° at B and C respectively. Then ABC is obviously the required triangle (Fig. 162).

Measurements :

Measure the sides CA and AB, and the angle BAC. What is the sum of the three angles A, B and C?

EXERCISE 4.11

Construct the triangle ABC in which :

1) $a = 4.5$ cm, B = 60°, C = 40°.
2) $b = 6.5$ cm, C = 38°, A = 70°.
3) $c = 5.2$ cm, A = 84°, $b = 4.8$ cm.

In each of the above cases, (1) measure the remaining sides and angles, and (2) find A + B + C.

1.47. By now it will have been noticed that the sum of the angles of any triangle is equal to π i e., 180°. This enables us to construct a triangle ABC, given any two of its angles, and one of its sides

The case in which the given side happens to be the included side has already been considered.

Case 4:—Let it be required to construct the triangle ABC given C, A and a.

Here the given side a is included not between the vertices C and A, but between B and C (Fig 163) However, we can calculate B, for

$$A + B + C = \pi$$
$$\therefore \quad B = \pi - A - C$$

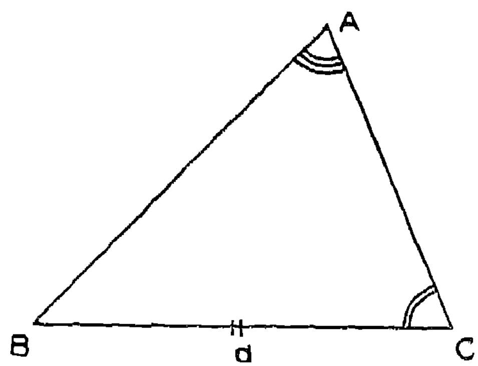


Fig. 163.

Thus in the triangle ABC, we know the angles B and C, and the included side a. Following the procedure detailed in Case (3) (§ 4.16), we can construct the triangle ABC.

EXERCISE 4.12

Construct triangle ABC given that ·

1) $a=5.2$ cm, $C=63°$, $A=40°$.
2) $b=6.5$ cm, $A=86°$, $B=62°$.
3) $c=4\ 8$ cm, $B=72°$, $C=64°$.

EXERCISE—4 13

Miscellaneous

1. Draw any segment AB and divide it into five congruent parts.
2. Take a segment OA on a given line. Take it as the unit segment. Find a point D on the line so that $|OD| = 3\frac{3}{7}$ units.
3. Let ABC be any triangle, and let D, E, F be the middle points of BC, CA and AB (Fig. 164)

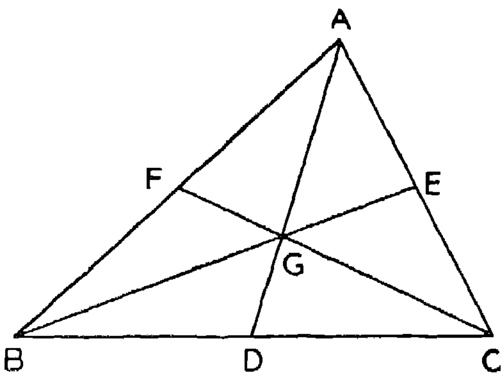


Fig. 164.

Now AD is the join of the vertex A to the middle point D of the opposite side BC It is called a *median* of the triangle ABC. It is the median bisecting the side BC. Similarly BE is the median bisecting the side CA and CF is the median bisecting AB.

verify that the three medians of any triangle are concurrent.

If G is the point of concuerrence, verify by using the dividers that AG = 2 GD, i.e.,

$$\frac{|AG|}{|GD|} = 2$$

What are the values of $\frac{|BG|}{|GE|}$ and $\frac{|CG|}{|GF|}$?

4. Draw a triangle ABC. Find D, the middle point of BC. Divide the median AD at C so that AG = 2/3 AD.

Let the line BG intersect AC at E, let CG intersect AB at F. Compare CE with AE and BF with AF.

EXERCISE—4.14

Miscellaneous—II

Section A:

1. (a) Draw $\angle ABC = 47^\circ$, bisect it.
   (b) Draw $\angle PQR = 63^\circ$, bisect it
   (c) Draw $\angle XYR = 123^\circ$, bisect it.

Section B:

1. Draw a triangle ABC: $|AB| = |AC| = 3''$, $|BC| = 2''$.
   (a) What sort of a triangle is ABC?
   (b) Bisect $\angle A$ and let the bisector meet BC at the point X. Is X the mid-point of BC?
   (c) Is the triangle symmetrical about AX?
2. Construct triangle XYZ in which $|YZ| = 4$ inches, $|ZX| = 3 5$ inches $|XY| = 3$ inches Bisect the angles of the triangle.
3 Draw a triangle with its sides 4 cm, 4.2 cm. and 5.8 cm. Will one of its angles be 90°?
4. (a) Draw accurately an isosceles right angled triangle with its base 3 inches long.

   (b) Draw accurately a triangle ABC with each side 8 cm. long Draw $AP \perp BC$, so that $P \in BC$ Measure AP.
5. Draw a triangle with its sides in the ratio 3 : 4 : 5. verify that one of its angles is a right angle
6. Draw a triangle with one side $5''$ long and the angles at the ends of the side 52° and 38°. Measure the other sides as accurately as you can.
7 The angles and sides of a triangle are as follows:—

   $A = 88^\circ$, $B = 40^\circ$, $C = 52^\circ$
   $a = 10 2$ cm, $b = 5.5$ cm, $c = 6.3$ cm.

Construct the triangle in three different ways by selecting appropriate data. Cut out the triangles and compare them.

8. In a triangle ABC, $\angle B = 25^\circ$, $c = 9.3$ cm $b = 3.8$ cm. Construct the triangle and show that there are two solutions.

Section C:

1. In the Isosceles triangle in which the angle at the vertex is (a) 45°, (b) 110°, (c) 90°, (d) 36°, find the remaining angles.
2. The congruent sides of an isosceles triangle are produced and each of the exterior angles so formed is 120°. Find the angles of the triangle.
3. Construct a square of side 3 cm.
4. Construct a parallelogram whose diagonals are 3 8 cm. and 5cm and one of the angles between them is 60°.
5. Construct a parallelogram ABCD where $|AB| = 2''$, $|BC| = 1.5''$ and diagonal $|AC| = 2.5''$. What particular form of parallelogram is it ?
6. Construct a rhombus whose diagonals are 6 cm. and 8cm. Measur its sides.
7. Given that (6 ; 6 ; 4) is an Archimedean solid with 24 vertices, here is the method of calculating the number of its faces and edges 2 hexagons and a square meet at each vertex. Hence there are 3 edges meeting at each vertex. It has 24 vertices. The total number of edges meeting at the vertices is $24 \times 3 = 72$. But in this method of enumeration, each edge is counted twice i e., from both its ends. Therefore, the solid has 36 edges Let us count the number of hexagonal faces. 2 hexagons meet at each of its 24 vertices. By counting like this each hexagon is counted at each one of its 6 angular points. Hence the number of hexagonal faces $= \dfrac{24 \times 2}{6} = 8$.

   Similarly it has $\dfrac{24 \times 1}{4} = 6$ square faces.

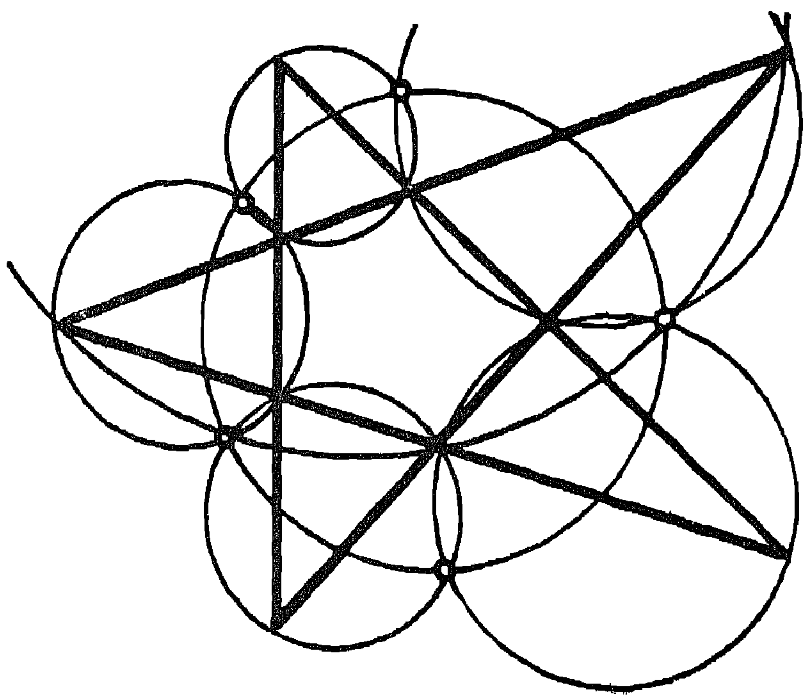
   In the same way compute (a) the number of edges and, (b) number of faces of each type in the following solids :—

   1. (4, 4, 4) with 8 vertices.
   2. (3, 3, 3, 3) with 6 vertices.
   3 (3, 4, 3, 4) with 12 vertices.
   4. (5, 5, 5) with 20 vertices.

5. (3, 3, 3, 3, 3) with 12 vertices.
6 (3, 5, 3, 5) with 30 vertices.
7. (6, 6, 5) with 60 vertices.
8. (3, 3, 3, 4) with 8 vertices.
9. (6, 6, 3) with 12 vertices.
10. (3, 3, 3, 3, 4) with 24 vertices.

Try to build card board models of these solids, with the help of your teacher

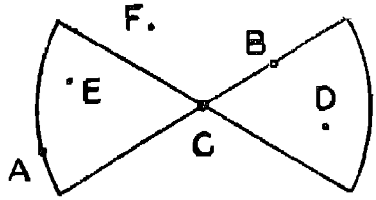
8 The vertices of a regular hexagon are joined to its centre. How many equilateral triangles are formed? How many rhombuses are formed?



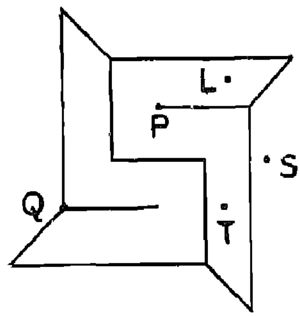
TEST PAPER 1

Into how many regions does each of the following figures divide the plane?

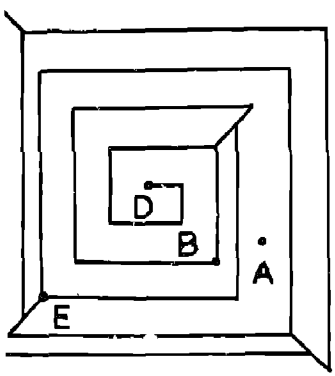
Colour different regions differently. Classify the marked points.



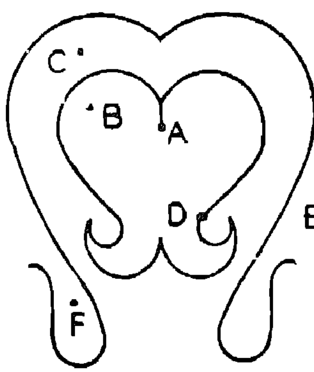
(a)



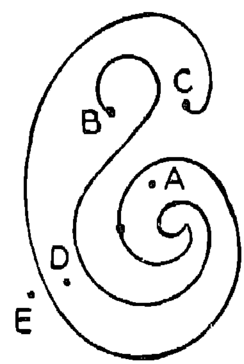
(b)



(c)

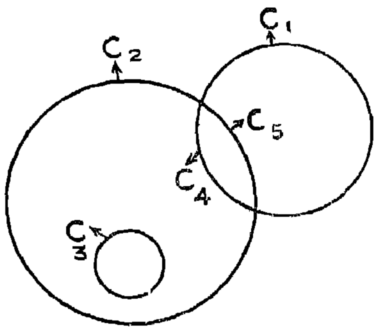


(d)

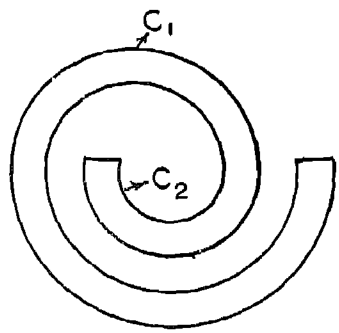


(e)

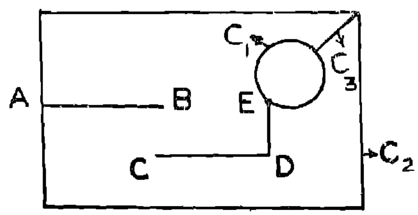
2. Examine the number of regions into which figures *a*, *b*, *c* and *d* divide the plane. Colour them differently and name their boundaries :—



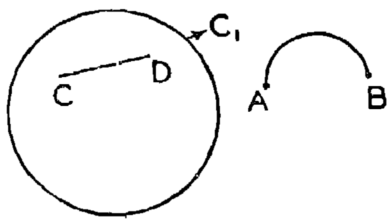
(a)



(b)



(c)



(d)

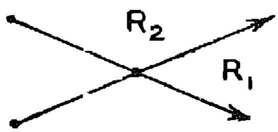
3. (a) Fill in the blanks correctly.

   1. Two parallel lines divide the plane into ——— convex regions.
   2. A pair of intersecting lines divide the plane into ——— regions.
   3. The figure that divides the plane into two regions both of which are convex is ———
   4. The set of all boundary points of a region is called ——— of the region.
   5. A segment joining any two boundary points of a convex region belongs to ———

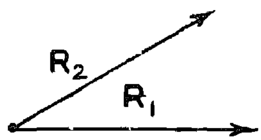
(b) State whether the following statements are true or false:

1. A connected set of points forms a region.
2. The set of points of a region is a connected set.
3. Every point of a region is an inner point.
4. Any point on the common boundary of two regions belongs to both the regions.
5. A point belonging to one region can also belong to another region.
6. The common part of two convex regions may not be a convex region.
7. A number of mutually parallel lines in a plane divide the plane into a number of convex regions.
8. A number of mutually parallel rays in a plane divide the plane into a number of convex regions.
9. A number of concentric circles divide the plane into a number of regions which are not convex.
10. If P is a point belonging to a convex region, a ray with P as its initial point crosses the boundary in two points.

. Classify the following as convex and non-convex regions:—

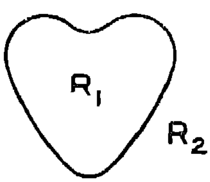


(a)

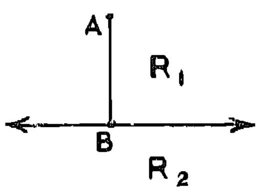


(b)

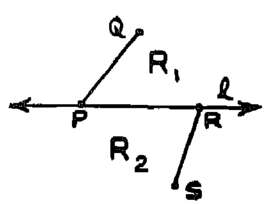
(c)



(d)



(e)



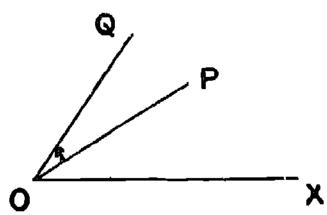
(f)

5. (a) Draw a quadrilateral whose interior is not convex. Draw its diagonals.
   (b) Draw a Jordan-curve and shade the inner region.

TEST PAPER 2

1. BE and CD are two segments such that BE $\cap$ CD=P. AP is parallel to EC. The orientation of angle APD is marked. Observe the figure carefully, mark the orientations of the other angles in the figure and fill in the blanks correctly.—
   1. The two transversals of the parallel lines are ——— and ———
   2. ——— and ——— are alternate angles.
   3. ——— and ——— are corresponding angles.
   4. The angle vertically opposite to $\hat{BPC}$ is ———
   5. The adjacent angles of $\hat{EPA}$ are ——— and ———
   6. The angle congruent to $\hat{APD}$ is ———

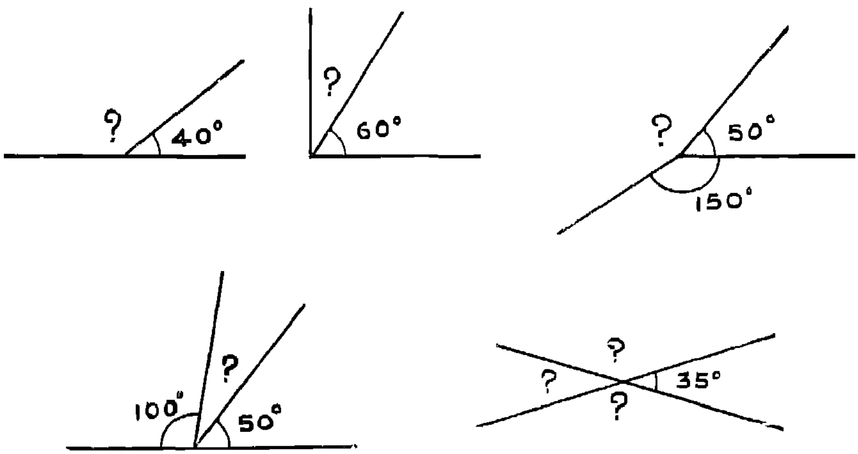
2. $\hat{POQ}$ is a given oriented angle and OX is a ray through O. Give a method of constructing an angle XOR congruent to $\hat{POQ}$.



TEST PAPER 3

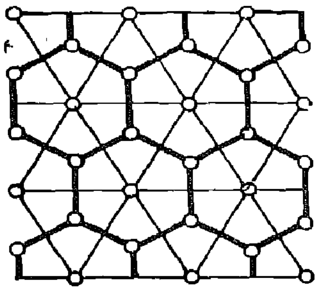
1. *a*, *b* are two given oriented segments and *l* is a given line. On *l* construct segments congruent to $a-b$ and $a+b$.

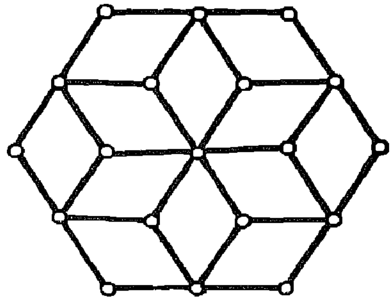
2. Fill in the blanks correctly :—
   1. If P, Q, R, S are four points marked in any order on a line $\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RS}$ = ————
   2. If $\overrightarrow{CD}$ is a segment and m, n are positive numbers, $m\ \overrightarrow{CD}+n\ \overrightarrow{CD}$ = ————
   3. The angle between the hands of a clock at 2 o'clock is the same as that at ———— o' clock with opposite orientation.
   4. The measure of the angle traced by the minute hand in 10 minutes is ———— degrees.
   5. 45° = ———— radians
   6. $\frac{3\pi}{4}$ = ———— degrees.
   7. Two adjacent angles whose sum is ———— degrees form a linear pair.
3. In the following figures find out the angles marked "?"

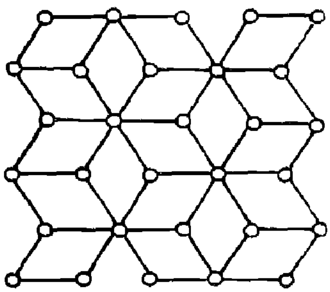


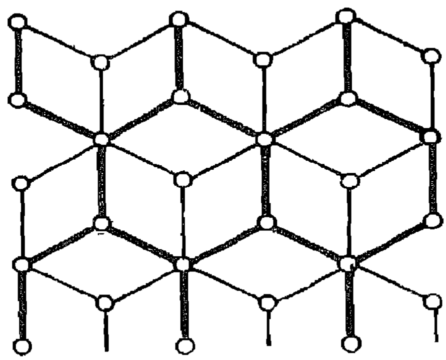
TEST PAPER 4

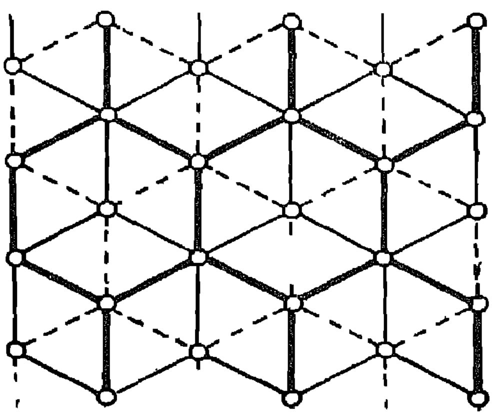
1. Construct an equilateral triangle ABC of side 4 cm. Draw the bisector of angle A to meet BC in D Measure BD, CD and angle ADB (use ruler and compasses only)
2. Using only ruler and compasses construct triangle ABC given AB = 6 cm, $\hat{B}=45^\circ$, $\hat{A}=45^\circ$. Draw the perpendicular from C to AB to meet AB at D. Measure $\angle$DAD, BD and CD.

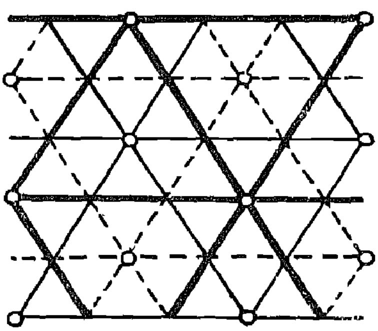


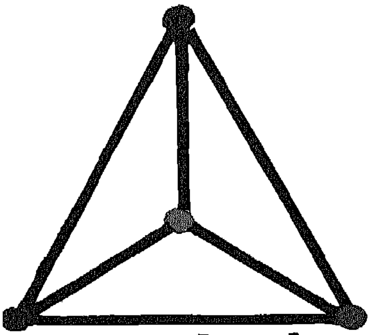




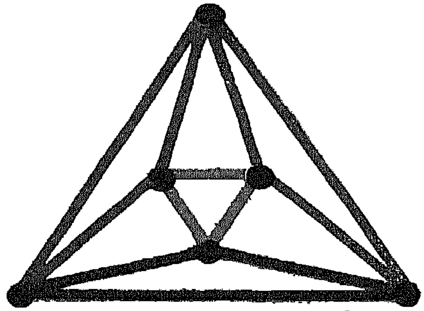




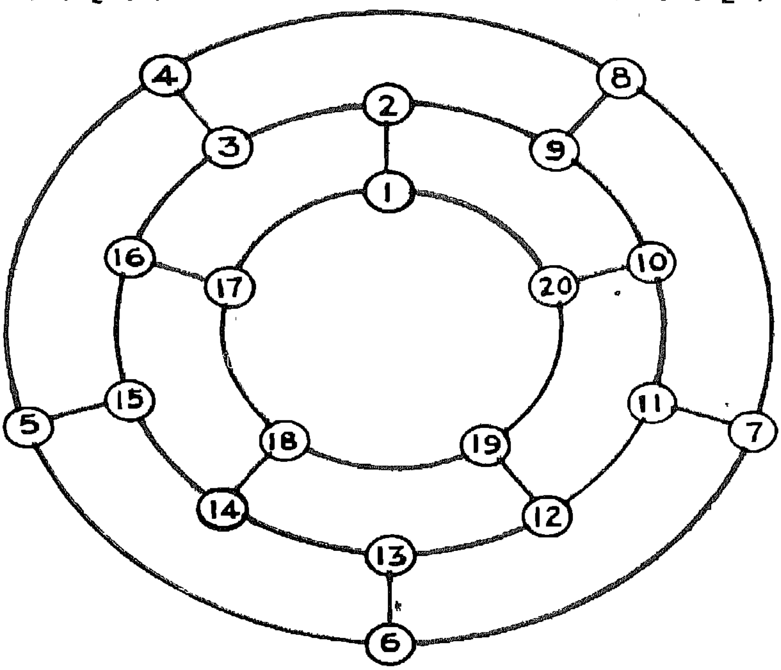




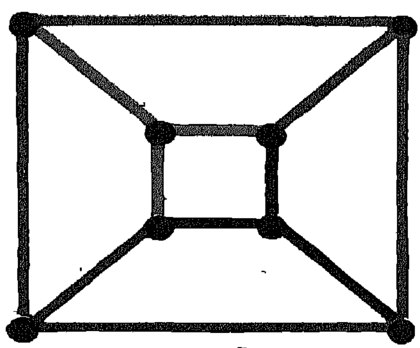
1·19 (a) [3,3,3]



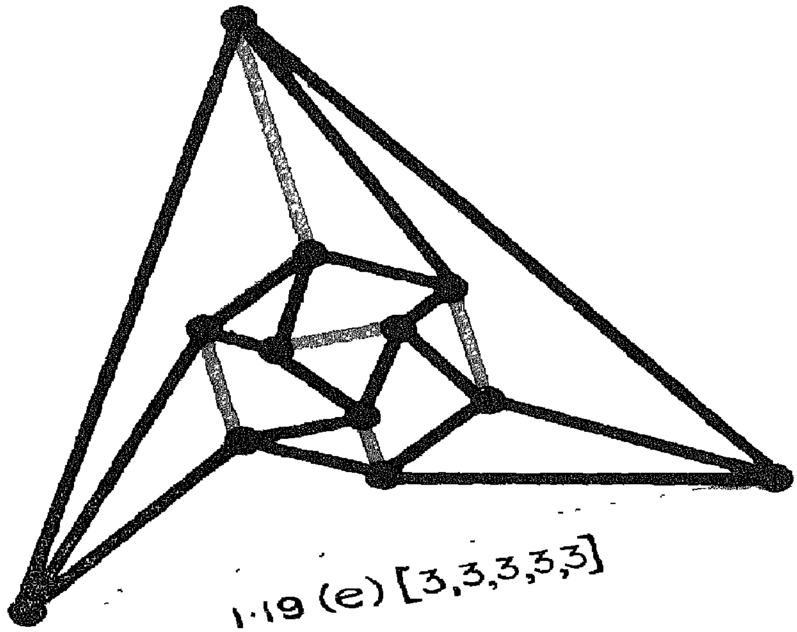
1·19 (c) [3,3,3,3]



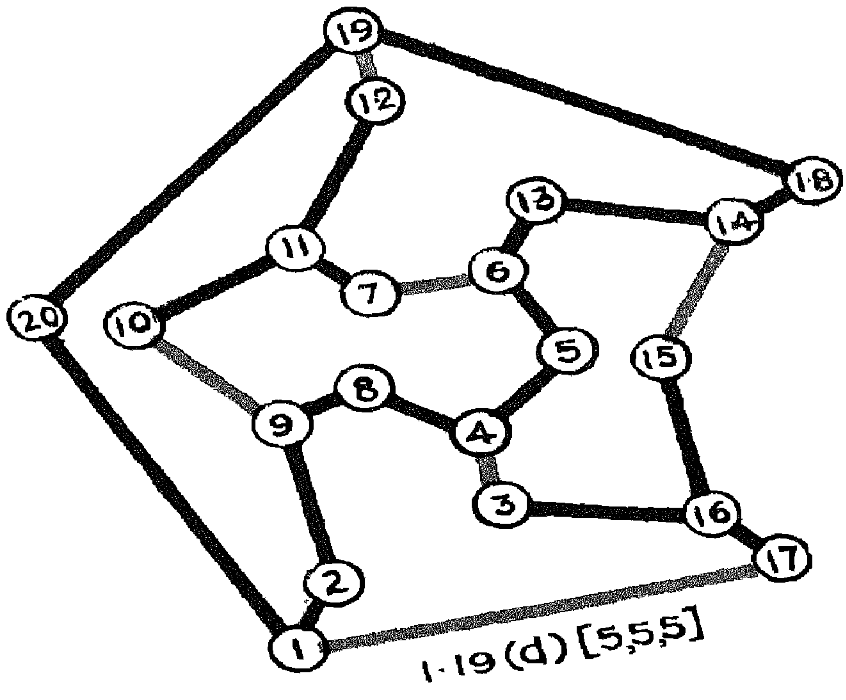
1·20



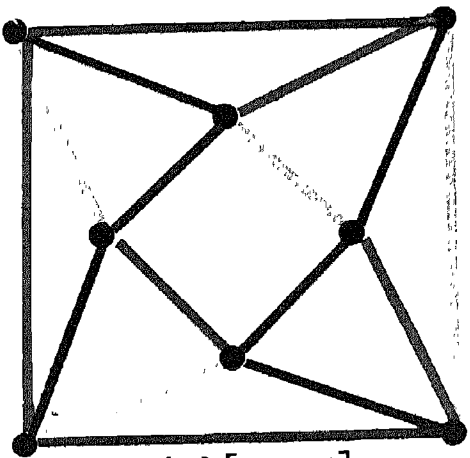
1·19 (b) [4,4,4]



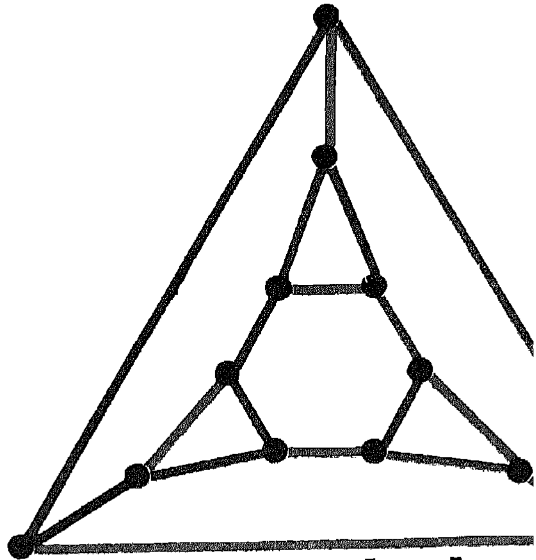
1·19 (e) [3,3,3,3,3]



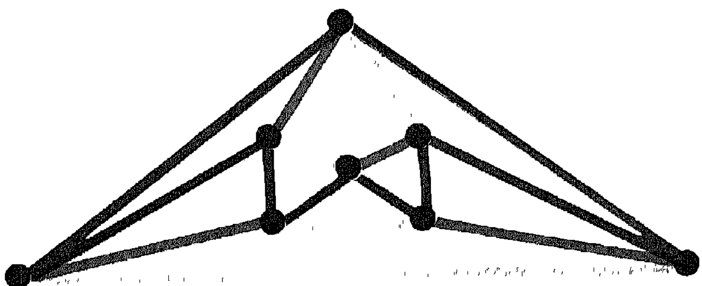
1·19 (d) [5,5,5]



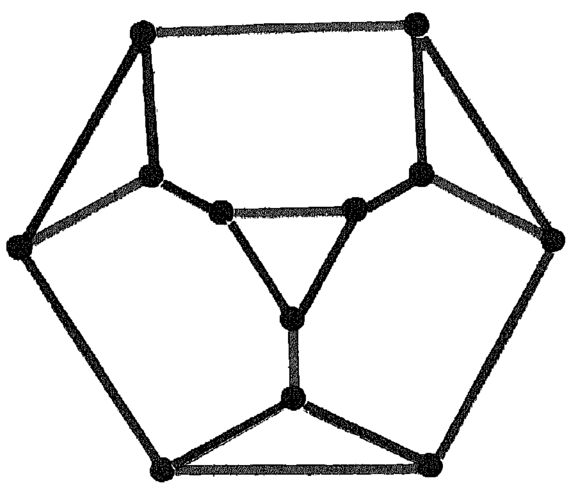
1·21 (a) [3,3,3,4]



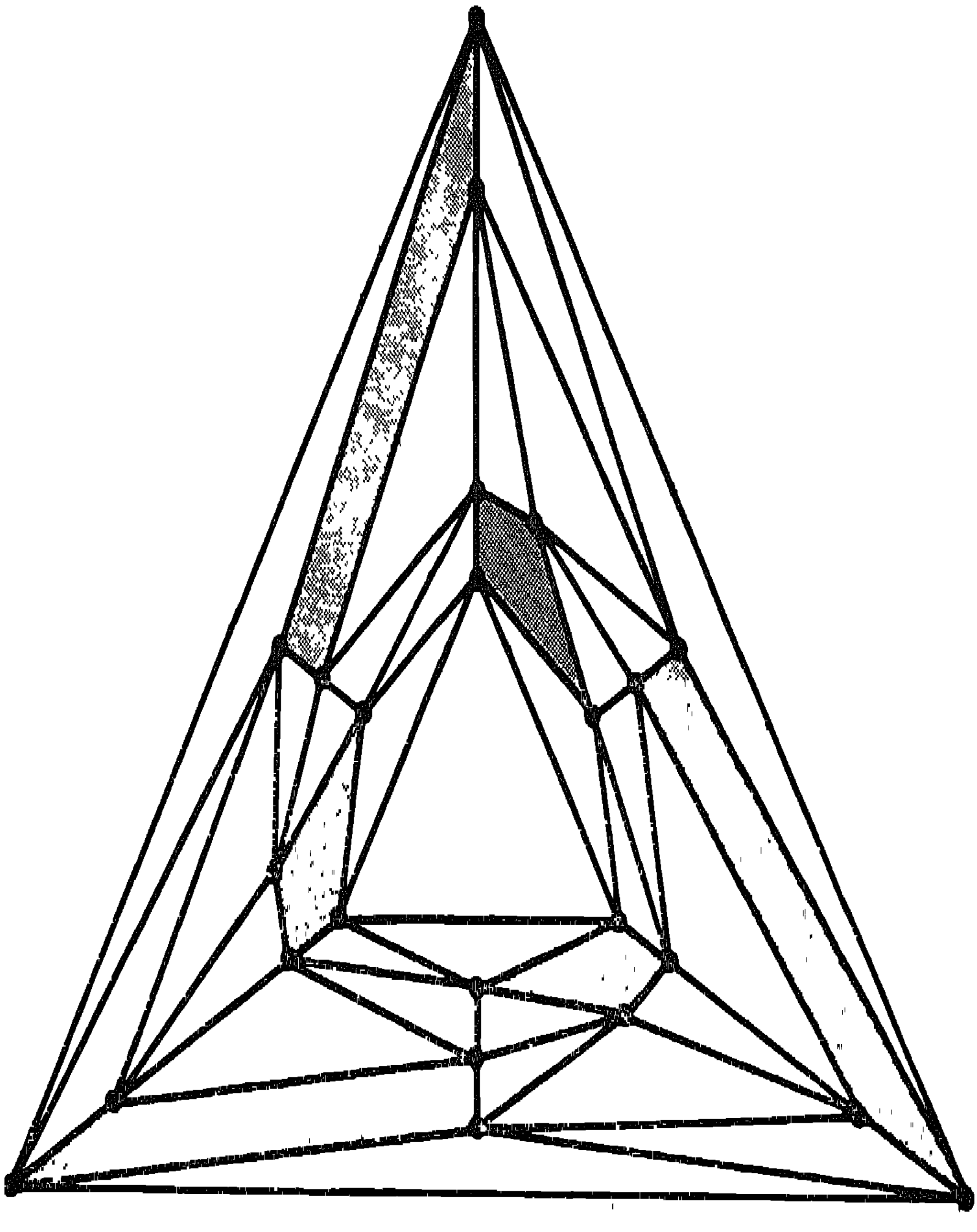
1·22 (a) [6,6,3]

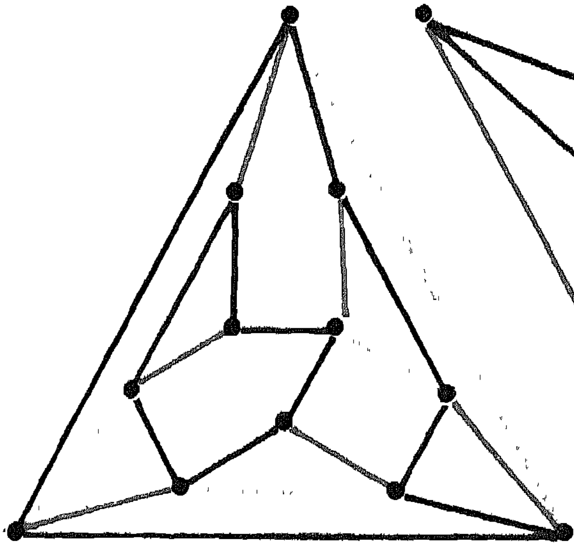


1·21 (b) [3,3,3,4]



1·22 (b) [6,6,3]





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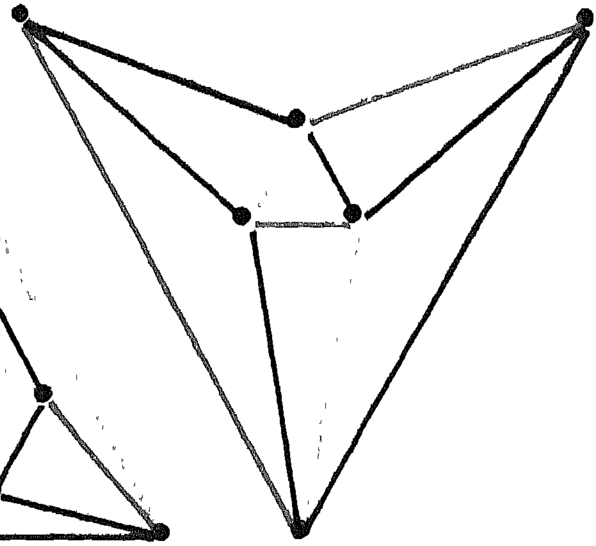

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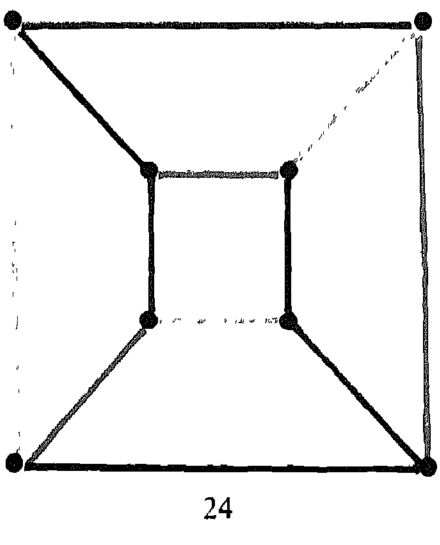

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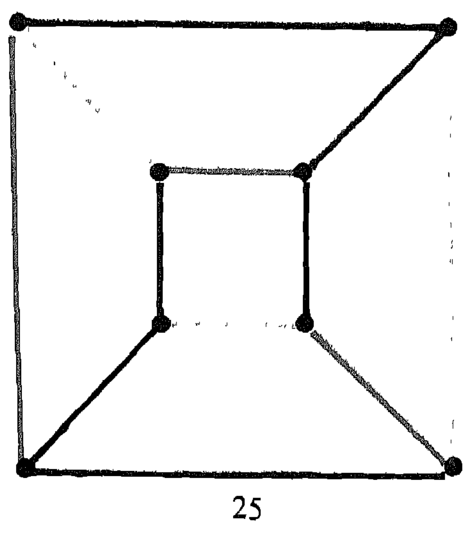

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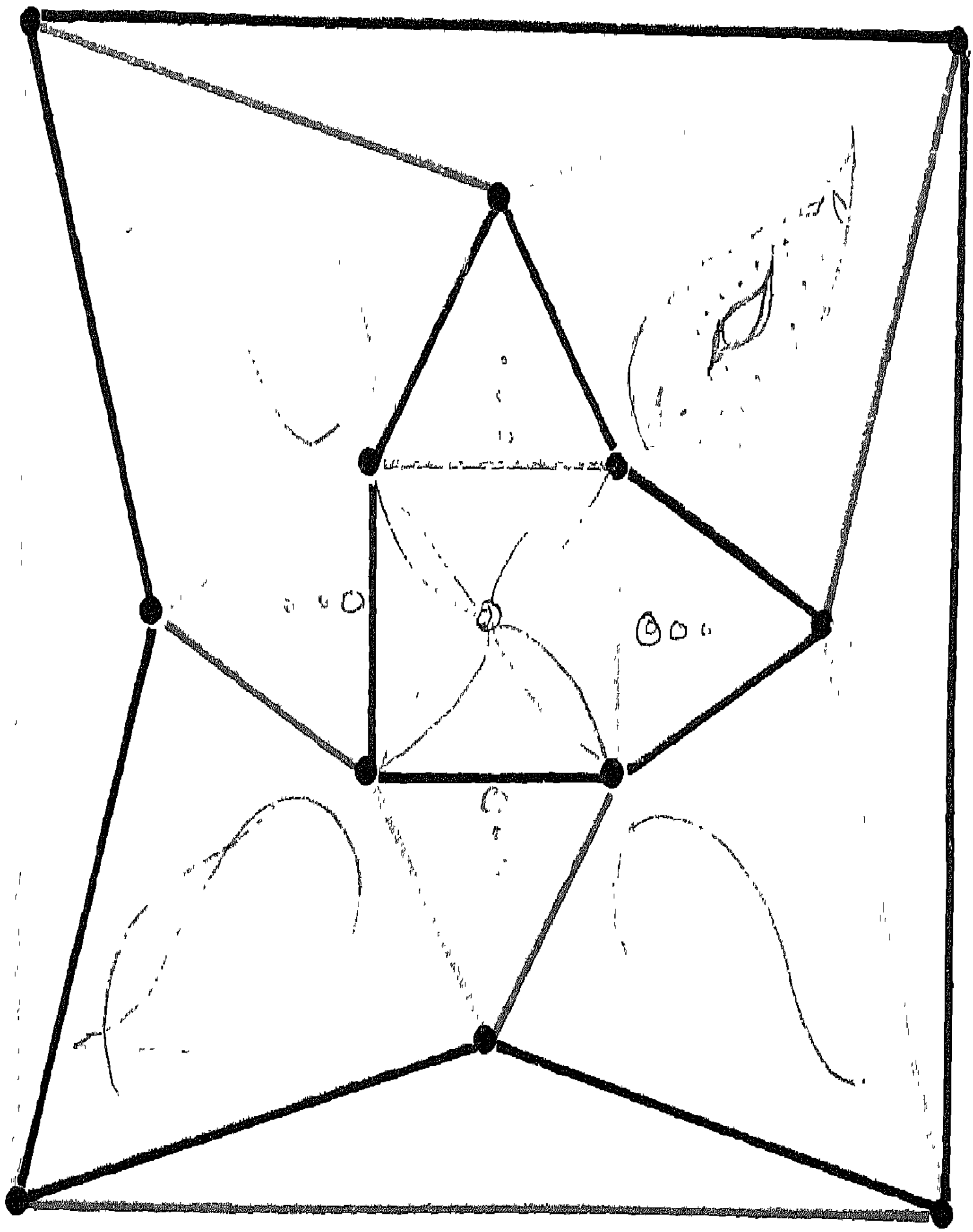

SOME INTERESTING PROBLEMS

Here is a list of miscellaneous interesting problems. Some of these can be done easily by the students. It is not intended that all these problems are to be solved by the students. Even if a student is unable to solve some problem of this set, he may obtain several steps towards the solution of the problem as explained in the problem itself The full solution of problems which interest a student may be obtained by him at any subsequent stage.

1. We say that a triangle PQR is inscribed in a triangle ABC if the vertices P, Q and R are on the sides (produced both ways if necessary) BC, CA and AB of the triangle ABC respectively. Draw any triangle ABC and a triangle PQR inscribed in it (Fig. 1). Take *any* point X on QR. Let BX and CX meet PQ and PR at Z and Y respectively. What do you find about the points A, Y, Z? Repeat this construction four or five times by taking several triangles. You find that

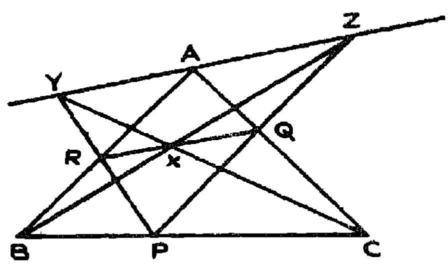

Fig. 130.

(i) $\triangle$ PQR is inscribed in $\triangle$ ABC,
(ii) $\triangle$ XYZ is inscribed in $\triangle$ PQR,
(iii) $\triangle$ ABC is inscribed in $\triangle$ XYZ,

The result of this theorem is due to the ancient Greek Geometer **Pappus.** A generalisation of this fundamental incidence theorem was discovered by the young brilliant genius **Pascal** in France in the 17th century when Pascal was a lad of twelve).

2. Draw a circle and take two sets of 3 points on it, say, A,B,C and A′,B′,C′, roughly as in Figure 2. Let X,Y,Z, be the

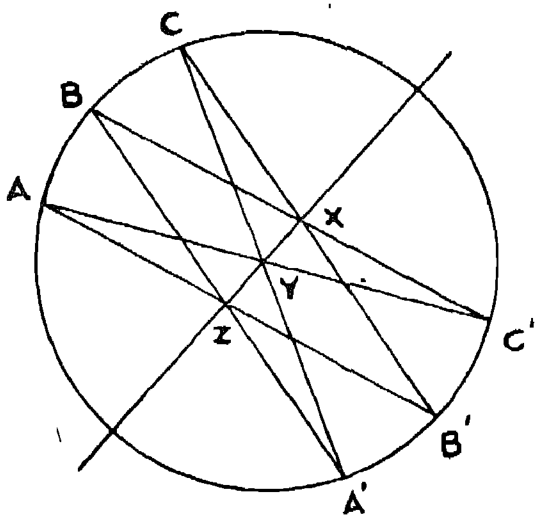

Fig 2.

intersections BC′ $\cap$ B′C etc. as in the figure; verify by drawing different circles and taking the six points in arbitrary ways, that X,Y,Z, always lie on a line.

3. You have been taught how to find the middle point M of the segment joining two given points *1* and *2*, utilising *ruler* and *compasses* (the basic Euclidean instruments). If the points *1* and *2* are marked on the sheet of paper it is possible to find their middle point M by utilising the compasses *only* in the following way.

Let the circles *1* and *2* drawn with centres *1* and *2* and passing through *2* and *1* respectively, intersect at the points *3* and *4*; draw the circle *3* with the point *3* as centre and passing through the point *4*, and let this circle intersect circle *2* (again) at the point *5*.

Let circle *5* whose centre is the point *5*, and which passes through the point *1* intersect circle *1* at the points *6* and *7*. Draw the circles *6* and *7* whose centres are the points *6* and *7* respectively, and both of which pass through the point *1*; these intersect (again) at the required middle point M. Verify this by using the ruler and dividers.

Draw all circles in this construction completely. Count the number of regions into which the plane is divided and mark those of them which are convex.

4 Suppose we have 9 coins, all looking alike; all of them are of the same weight except one which is lighter than the rest being a false coin. Only a balance is given (sets of weights are not provided) and you are asked to employ the balance only *twice* and find the false coin. There is also another restriction, viz., that the weighings must be done by putting 3 coins in each of the pans when you use the balance. In order to solve the problem we use the geometric figure consisting of a square and its middle lines of symmetry as shown in the accompanying figure (Fig. 3).

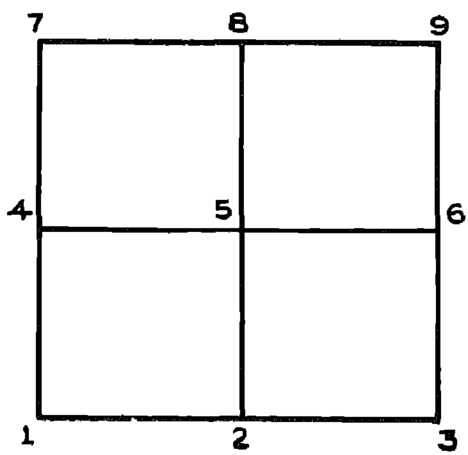

Fig. 3.

Let the coins be marked with numbers *1* to *9*. Let the first weighing be done by putting coins (*1,2,3*) in the left pan, and coins (7,8,9) in the right pan; if the *left pan* becomes heavy, then the false coin is evidently in the coins corresponding to numbers in the topmost line

of the square; if it becomes light, then in the bottommost line, and if the two pans balance, the false coin must be among those in the middle line parallel to the top and bottom lines. Name the coins to be weighed in the second weighing by examining the square. Using the fact that a pair of non-parallel lines in the figure intersect exactly in a point, explain how you can find the false coin in this way.

5. Can you give a method of solving the problem below? It is supposed to have occurred to an engineer who was in charge of a government campus.

A number of sites are to be allotted to the ministers and their secretaries of a State Administration. The sites (which are sufficiently large in number) are situated just outside the circumference of a disc. The government consists of one Prime Minister who is to have 3 secretaries, say, and 2 cabinet ministers who have 2 secretaries each, and 12 other ministers, who have 1 secretary each. The ministers have chosen their sites. These sites are shown

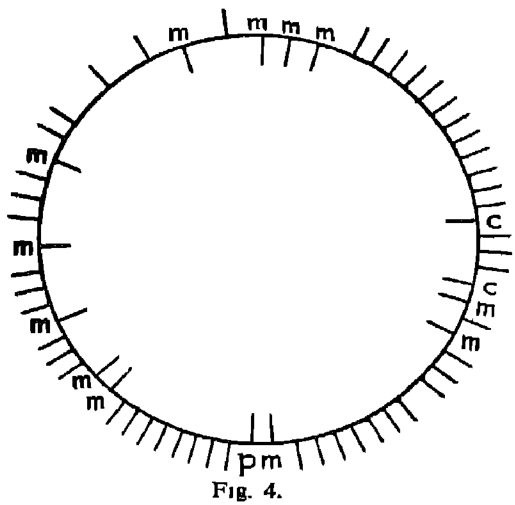

Fig. 4.

by line segments running inside the disc marked by the letters p,c,m, to designate the Prime Minister, cabinet ministers and the ministers and the empty sites are shown by line segments outside the disc (Fig. 4). The engineer has to allot the sites for the respective secretaries and mark roads connecting the residence of each minister to his secretary (or secretaries) which run inside the disc so that no two roads cross each other. Solve the problem in such a way that even if 2 or 3 ministers are added to the

ministry and sites are allotted to them later, one can make roads with the same conditions and without changing the roads already made.

6. Suppose you take a rupee coin and trace its contour to get a circle. Suppose we want to find the centre of this circle and you are allowed to use only the compasses and not the ruler. One way of finding the centre (the method is due to the 18th Century Italian Mathematician **Mascheroni** is described below). Do the construction by using a rupee coin or drawing a circle (not marking its centre) and verify according to this method that you actually get the centre

   Construction: Take any point A on the circle; with A as centre and any radius r less than the diameter of the given circle and greater than $\frac{1}{4}$ of its diameter, draw a circle cutting the given circle at B and C. Let the circle with B as centre and BA as radius cut the second circle at D and E. Again let the circle with D as centre and DE as radius cut the second circle again at F. With centres A and F and radii $= |FC|$ draw circles cutting at P (say). Let the circle with P as centre

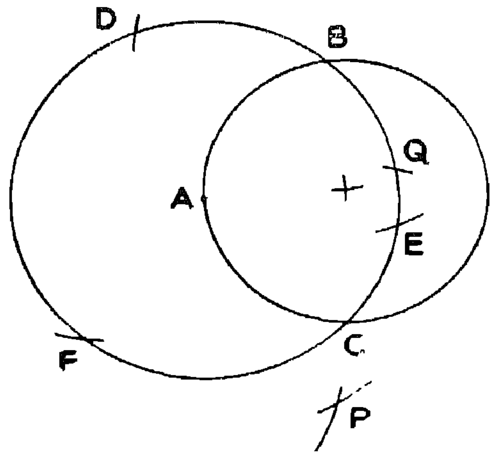

Fig. 5.

and $|PA|$ as radius, cut the second circle at Q Then the circles with A and B as centres and BQ as radius, cut at the centre of the first circle. You must do the construction after studying the figure above, where the appropriate points of intersection of the circles are indicated. (This is to test the accuracy of your using the compasses).

7. You have learnt the concept of a region in a plane. We can define regions even on curved surfaces such as a sphere (eg: surface of a ball on the earth) or an anchor-ring (tennicoit ring) surface in an

analogous way. (Connectivity will be by paths which lie on the surface, and a *dot* put around an inner point will be a corresponding *dot* on the surface). These surfaces have not got any boundary point at all. (Their boundaries are *empty sets*). Suppose we draw a Jordan-curve on the sphere; it divides the sphere into two regions. Can you draw a Jordan-curve on the anchor-ring so that it does not divide the surface? Show that you can draw two such curves which do not divide its surface at all. (Make an experiment with a tennicoit-ring or any similar surface.)

8. (a) Take a long strip of thin transparent plastic material in the form of a rectangle; bend and paste its opposite smaller edges so that we get the curved surface of a cylinder (fig.6).

(We consider points lying on opposite sides, that is, the facing side and the reverse side of the sheet, as a single point—i.e., the thickness of the sheet is ignored) We obtain a single region. What is its boundary? How many distinct connected pieces does it consist of?

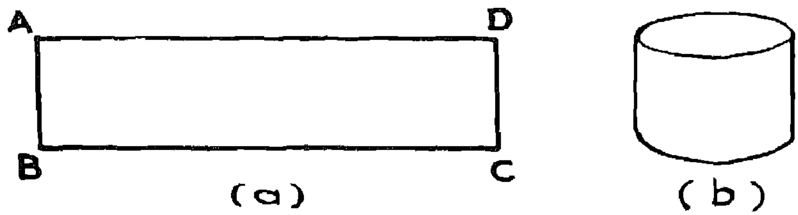

Fig. 6. (a) and (b)

Suppose you draw a Jordan-curve on this. Does it divide it into two regions always?

(b) Take again a similar strip and paste its edges after twisting it by a half-turn (Fig. 7-1).

The surface you get is called a *Mobius band* (in honour of the great nineteenth century German Geometer **Mobius**) The thickness of the strip is ignored so that points lying on opposite sides of the sheet are considered as the same point, i. e., you may take the points on it as small holes made by a pin. You again obtain a region What is its boundary and how many distinct unconnected pieces does it consist of? Suppose you draw the Jordan-curve which goes in the middle of the strip; does it divide the surface into two regions? Show that it is not so and we have again a single region. Cut the strip along the middle and verify that when you open it out you have a single piece and not two distinct pieces.

If an ant were to traverse the middle line observe that when it comes back its left-right orientation is reversed We therefore find that

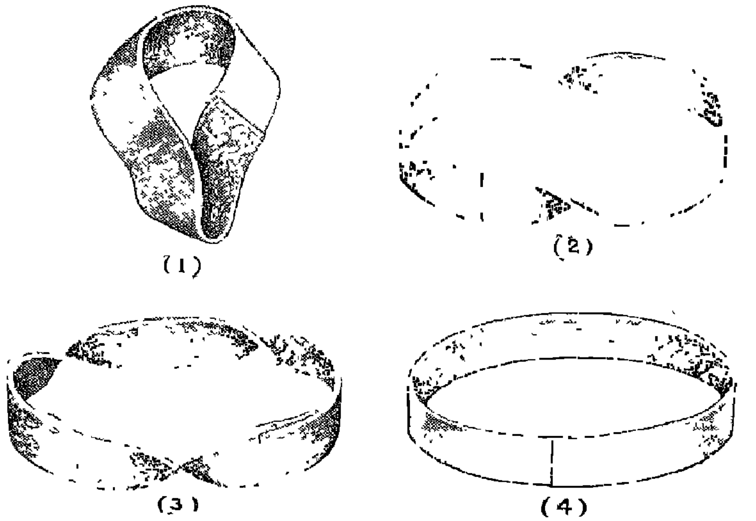

Fig. 7.

we cannot have the same orientation on a Mobius band or that it is non-orientable, in contrast to a plane where we can have the same orientation Take a Mobius band of constant width Cut it along a line which runs at a distance of one-third of the width of the band from the boundary and see what you get

Make other bands by pasting the strips after two half-twists (Fig. 7-2) and after three half-twists (Fig. 7-3) See what happens when we cut them across the middle line and the trisection line.

(c) Take a small square *thin* transparent plastic sheet and cut out and remove two rectangles out of it as shown in Fig. 8. Cut along the lines AB, CD, EF, getting two pieces. Paste the upper strip on the lower along the cuts AB, CD and EF, after a half-twist (all in the same way) along each of the three edges. You now get a surface which was constructed by the great Russian Mathematician of the present century, **Pontryagin.** This surface has several interesting features It is orientable and has a single boundary piece which is in the form of a

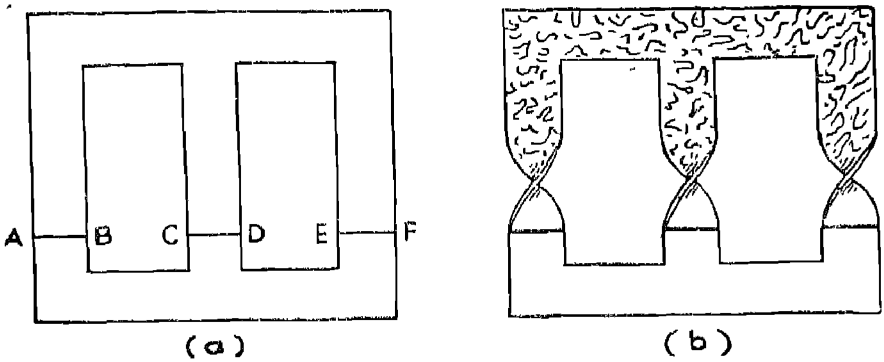

Fig 8.

simple knot. Keep a plastic insulated wire along its edge and join its ends and verify that you get a knot.

9. Suppose you are given a square sheet of paper of edge-length 15 cm, say, and you are asked to make *cuts* (only) in it so that

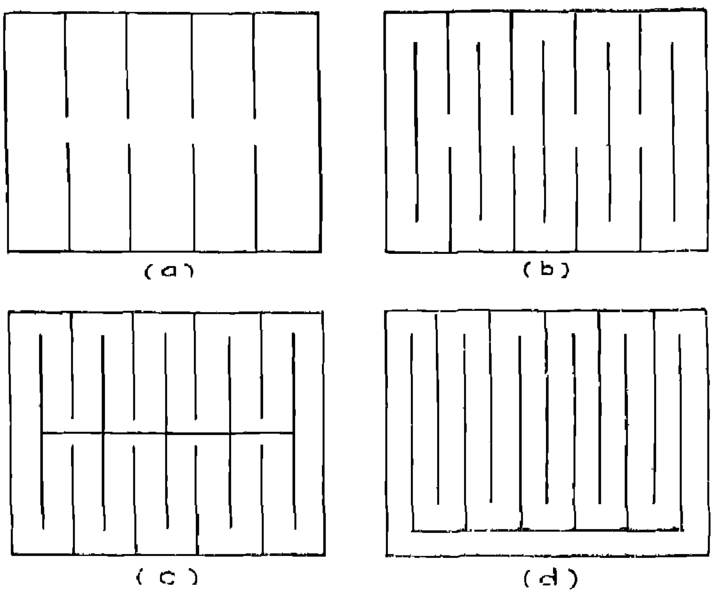

Fig. 9.

when it is spread out, it is possible for your body to pass through the hole made in it. You may first of all think it is not possible, but it *is* possible and the method of making the cuts and the

reason why it is possible are explained below. You may answer the following questions in turn.

(a) How many regions are there when we draw lines in the sheet as in Fig. 9 (a)? How many distinct connected pieces does its boundary consist of?

(b) Answer the questions with reference to each of the figures 9 (b) and 9 (c) Make cuts as in Fig. 9 (c) spread out and verify that your body can pass through the hole so formed Can you tell the reason?

(c) You can cut it along the lines shown in figure 9 (d) for the same purpose. Try it out.

10. You have learnt (Article 2·1, p 50-51) that the orientation obtained at the same point by following different segment chains will be the same.

This property can be verified as explained below.

Let us adopt the convention that the orientation at the point P is such that every ray from the point is to be crossed from the unshaded side to the shaded side of the ray.

On a graph sheet mark the following points.

$P \equiv (1, 1)$,	$A_1 \equiv (1, 2)$,	$A_2 \equiv (4, 2)$,	$A_3 \equiv (4, 1)$
$A_4 \equiv (3, 1)$,	$A_5 \equiv (3, 4)$,	$Q \equiv (4, 4)$,	$R \equiv (5, 5)$
$B_1 \equiv (2, 1)$,	$B_2 \equiv (2, 4)$,	$B_3 \equiv (1, 4)$	
$B_4 \equiv (1, 3)$,	$B_5 \equiv (4, 3)$,		

Shade the segment PA_1 on that side of it which contains A_3. This determines that A_1A_2 is to be shaded on that side which contains P. A_3 is on that side of $A_1 A_2$ on which it is shaded. Hence $A_2 A_3$ is to be shaded on that side of it which contains A_1. Continuing like this along the segment chain $PA_1 A_2 A_3 A_4 A_5$ QR, shade one side of the ray QR.

Next, starting from P, and following the segment chain P B_1 B_2 B_3 B_4 B_5 QR, determine the mode of shading the ray QR and verify that the two different paths lead to the same orientation at Q.

Verify this global property of the plane for the following two paths obtained by joining these points in order:

(a) (0, 0), (2, 1), (−6, 9), (−5, 10), (−4, 9), (−3, 10), (−2, 9), (−2, 11), (0, 8), (0, 12).

(b) (0, 0), (−2, 1), (6, 9), (5, 10), (4, 9), (3, 10), (2, 9), (2, 11), (0, 8), (0, 12)